Qualifying Exam – Partial Differential Equations II – May 2024

INSTRUCTIONS: ***The exam is closed-book***– *Notes, resources or communications are strictly prohibited*! You may use the well-known theorems or results without proof, but you must indicate such a result clearly with all necessary steps. You may also use the result of one problem to solve another problem on the exam.

In all the following, unless specified, Ω will denote a bounded open set in $\mathbb{R}^n$.

#1. (15 points) Let $p > 1$, $n \geq 2$ and $\Omega = \{x \in \mathbb{R}^n : 1 < |x| < 2\}$. Assume $v \in W_0^{1,p}(1,2)$ and let $u(x) = v(|x|)$. Show that $u \in W_0^{1,p}(\Omega) \cap C^{0,\gamma}(\bar{\Omega})$, where $\gamma = 1 - \frac{1}{p}$.

#2. (10 points) Let $B_R(a)$ denote the open ball in $\mathbb{R}^n$ with center a and radius R. Prove the inequality

$$\int_{B_R(a)\setminus \bar{B}_{R/2}(a)} (u - u_R)^2 dx \leq C_n R^2 \int_{B_R(a)\setminus \bar{B}_{R/2}(a)} |Du|^2 dx \quad \forall\, u \in H^1(B_R(a)),$$

where u_R is the average of u over $B_R(a) \setminus \bar{B}_{R/2}(a)$ and C_n is a constant independent of R and a.

#3. Let $u \in H^1_{loc}(\mathbb{R}^n)$ be a weak solution of equation $-\sum_{i,j=1}^n D^i(a_{ij}(x)D^j u) = 0$ in $B_R(0)$ for all $R > 0$, where the equation is assumed to be uniformly elliptic in all of $\mathbb{R}^n$.

(a) (10 points) Show that there exists a constant C such that for all $R > 0$ and $\lambda \in \mathbb{R}$,

$$\int_{B_{R/2}(0)} |Du|^2 dx \leq \frac{C}{R^2} \int_{B_R(0)\setminus \bar{B}_{R/2}(0)} (u - \lambda)^2 dx.$$

(b) (10 points) Assume $|Du| \in L^2(\mathbb{R}^n)$. Show that u is constant almost everywhere on $\mathbb{R}^n$.

#4. (15 points) Let $\beta \in C^1(\mathbb{R}^n)$ and $\alpha \in C^1(\mathbb{R})$ with $\alpha' \geq 0$. Show that given any f and g, the boundary value problem

$$\begin{cases} -\sum_{i=1}^n e^{x_i} u_{x_i x_i} + \beta(Du) + \alpha(u) = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

cannot have more than one solution in $C^2(\Omega) \cap C^1(\bar{\Omega})$.

#5. Let $\Phi \in C^1(\bar{\Omega}; \mathbb{R}^n)$ and $c \in C(\bar{\Omega})$. Consider the operator

$$Lu = -\sum_{i=1}^n D^i(e^{x_i} D^i u) + \Phi \cdot Du + cu.$$

(a) (10 points) Find the formal adjoint operator L^* of L on $H_0^1(\Omega)$ in the sense that the bilinear form B^* associated with L^* satisfies $B^*[u,v] = B[v,u]$ for all $u, v \in H_0^1(\Omega)$, where B is the bilinear form associated with L.

(b) (15 points) Show that $Lu = f$ has a unique weak solution $u \in H_0^1(\Omega)$ for each $f \in L^2(\Omega)$ provided that one of the following conditions holds:

(i) $c \geq 0$ in Ω. (ii) $c \geq \operatorname{div} \Phi$ in Ω. (iii) $c \geq \frac{1}{2} \operatorname{div} \Phi$ in Ω.

#6. (15 points) Let $n \geq 2$, $1 < p < n$, $0 < q < \frac{np+p-n}{n-p}$ and $q \neq p - 1$. Use constrained minimization to show that the boundary value problem

$$\begin{cases} -\operatorname{div}(|Du|^{p-2} Du) = |u|^{q-1} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a nontrivial weak solution in $W_0^{1,p}(\Omega)$ satisfying $u \geq 0$ almost everywhere in Ω.