PDE II: Qualifying Exam

Friday May 9, 2025

There are five equally weighted problems. Start with problems that are most familiar.

Question 1 *Fix $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^d$ open, connected, and bounded. Prove that if $u \in L^p(\Omega)$ satisfies*

$$Du = 0, \qquad a.e.\ in\, \Omega,$$

then $u \in W^{1,p}(\Omega)$ and there exists $c \in \mathbb{R}$ such that $u = c$ a.e. in Ω.

Question 2 *Fix $p \in [1, \infty)$ and $x \neq y \in \mathbb{R}$. Define the linear operator $T_{x,y} : \mathcal{C}^0(\mathbb{R}) \to \mathbb{R}$ by*

$$T_{x,y} u = \frac{u(x) - u(y)}{|x - y|^{\frac{1}{q}}},$$

where q is the Sobolev conjugate to p. Show that $T_{x,y}$ has an extension to $W^{1,p}(\mathbb{R})$ that satisfies

$$|T_{x,y} u| \leq \|u\|_{W^{1,p}(\mathbb{R})},$$

for all $u \in W^{1,p}(\mathbb{R})$.

Question 3 *Assume $F : \mathbb{R} \mapsto \mathbb{R}$ is $\mathcal{C}^1$ with F' bounded. If $\Omega \subset \mathbb{R}^d$ is bounded and open, then show that $u \in W^{1,p}(\Omega)$ for some $p \in [1, \infty)$ implies that $F(u) \in W^{1,p}(\Omega)$ and*

$$\partial_{x_i} F(u) = F'(u) u_{x_i},$$

for $i = 1, \ldots, d$.

Question 4 *Let $\Omega \subset \mathbb{R}^d$ be bounded, open, with a smooth boundary. Let $c(x) \geq 0$ be a smooth function. Show that there is a solution $u \in \mathcal{C}^2(\Omega)$ to*

$$\begin{aligned} -\Delta u &= \frac{1}{1 + e^{-u}} - c(x) u, && in\, L^2(\Omega), \\ u &= 0, && in\, L^2(\partial\Omega), \end{aligned}$$

that satisfies $u > 0$ on Ω.

Hint: $f(u) := (1 + e^{-u})^{-1}$ is an increasing, bounded function and $f(0) > 0$.

Question 5 *Consider the operator*

$$\mathrm{L}u := -\Delta u + b \cdot Du,$$

where $b \in \mathbb{R}^d$ *is a* **constant** *vector of norm 1 and* $\Omega \subset \mathbb{R}^d$ *is an open, bounded set with a* $\mathcal{C}^1$ *boundary and exterior unit normal* ν*. Let* $\partial\Omega$ *be divided into two disjoint, connected sets* Γ_1 *and* Γ_2 *each of which has positive surface measure.*
(a) For $f \in L^2(\Omega)$ *and* $\alpha \in \mathbb{R}$*, find a weak formulation for the problem*

$$\begin{aligned} \mathrm{L}u &= f, && \text{in } \mathrm{L}^2(\Omega), \\ u &= 0, && \text{in } \mathrm{L}^2(\Gamma_1), \\ \nu \cdot \nabla u + \alpha u &= 0 && \text{in } L^2(\Gamma_2). \end{aligned}$$

Show for your weak formulation that if $u \in H^2(\Omega)$ *and* u *is a weak solution then* u *is a strong solution.*
(b) Show that if $\alpha \geq \frac{1}{2}$ *then for each* $f \in L^2(\Omega)$ *your weak formulation in part (a) has a unique solution* $u \in H^1(\Omega)$*, with zero trace on* Γ_1*.*
