MTH849-S2020 QUALIFYING EXAM

1. Instructions

There are five problems on this exam. Complete as many problems as possible. Your four highest scoring answers will be used to determine your grade on the exam. A preference in scoring will be given to complete answers to entire problems, in contrast to partial answers to possibly more problems. Do not use any outside materials during this exam.

2. The exam questions

Problem 1. Assume that $\Omega \subset \mathbb{R}^n$ is open and bounded, and assume that $\alpha(x) \geq 0$ with

$$\int_\Omega \alpha(x)dx > 0.$$

Prove that there exists a constant, C, depending upon Ω and α so that the following holds:

$$\forall\ u \in H^1(\Omega), \quad \int_\Omega u^2 dx \leq C\left(\int_\Omega |\nabla u|^2\, dx + \int_\Omega \alpha(x)(u(x))^2 dx\right).$$

Problem 2. Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded, and non-empty, with $B_1(0) \subset \Omega$, and $n > 2$.

Of the following collection of pairs X and $\langle \cdot, \cdot \rangle_X$, exactly one is a Hilbert space. For those that are not Hilbert spaces, give an explanation as to why, and for the one that is a Hilbert space, prove that it is.

You are free to assume that you already know that $H^1(\Omega)$ and $L^2(\Omega)$ are complete in their usual norm (inner product). In particular, you may freely use that you already know that Cauchy sequences are convergent in the respective norms in $H^1(\Omega)$ and $L^2(\Omega)$.

You are also free to invoke Problem 1 as you see fit.

(i)

$$X = \{u \in H^1(\Omega) \cap C(\Omega)\ :\ u(0) = 0\}, \qquad \langle u, v\rangle_X = \int_\Omega \nabla u \cdot \nabla v dx.$$

(ii)

$$X = H^1(\Omega), \qquad \langle u, v\rangle_X = \int_\Omega \nabla u \cdot \nabla v dx.$$

(iii)

$$X = \{u \in H^1(\Omega)\ :\ a.e.\ x \in B_{1/2}(0),\ u(x) = 0\}, \qquad \langle u, v\rangle_X = \int_\Omega uv dx.$$

(iv) Let $\alpha(x) \geq 0$ and $\int_\Omega \alpha dx > 0$, and define:

$$X = H^1(\Omega), \qquad \langle u, v\rangle_X = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega \alpha(x)u(x)v(x)dx.$$

As a follow-up, when $n = 1$, in fact two of these spaces are Hilbert spaces. Which two are they? What has changed between $n = 1$ and $n \geq 3$? You do not need careful justification for the follow-up question.

Problem 3. Assume that Ω is a bounded, open domain with $\partial\Omega$ is C^1, and further assume that there exists a smooth vector field, $\boldsymbol{\alpha} : \mathbb{R}^n \to \mathbb{R}^n$ with the property that there exists $c > 0$ with $\forall\ x \in \partial\Omega$, $\boldsymbol{\alpha}(x) \cdot \boldsymbol{\nu}(x) \geq c$. (Recall, $\boldsymbol{\nu}$ is the outward normal to $\partial\Omega$.)

Use an integration by parts argument to prove that there exists C depending on Ω, n, p, and c

$$\forall\ u \in C^1(\overline{\Omega}), \qquad \int_{\partial\Omega} |u|^p\, dS \ \leq\ C \int_\Omega |Du|^p + |u|^p\, dx.$$

Give an example of Ω and $\boldsymbol{\alpha}(x)$ which satisfy the assumptions of this result.

Problem 4. Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded, with $\partial\Omega$ is C^1. The non-homogeneous Neumann equation is

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

for $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, and $u \in H^1(\Omega)$. We will say u is a weak solution of (1) if the following is true:

$$\forall\ v \in H^1(\Omega), \qquad \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx + \int_{\partial\Omega} g \mathrm{T} v dS, \tag{2}$$

where T is the trace operator $T : H^1(\Omega) \to L^2(\partial\Omega)$.

(i) Prove that if

$$\int_{\partial\Omega} g dS + \int_\Omega f dx = 0.$$

then (1) admits a weak solution.

(ii) Prove that if (1) admits a weak solution, u, then

$$\int_{\partial\Omega} g dS + \int_\Omega f dx = 0.$$

(iii) Is the solution in part (i) unique? Why or why not?

(iv) Next, consider a modified version of (1), which is given by

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{3}$$

We say that $u \in H^1(\Omega)$ is a weak solution of (3) provided that

$$\forall\ v \in H^1(\Omega), \qquad \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega u v dx = \int_\Omega f v dx + \int_{\partial\Omega} g \mathrm{T} v dS, \tag{4}$$

Prove that for all $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, there exists a unique weak solution of (3).

Problem 5. Assume that $\Omega \subset \mathbb{R}^n$ is open and bounded, with a C^1 boundary. We take as a definition that u is a weak solution of

$$-\Delta u + u = f \ \text{ in } \ \Omega,$$

if u satisfies

$$\forall\ v \in H^1(\Omega), \qquad \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega u v dx = \int_\Omega f v dx.$$

(i) Assume, without proof, that you already know that for all $f \in L^2(\Omega)$, there exists a unique $u_f \in H^1(\Omega)$ that is a weak solution of $-\Delta u + u = f$ in Ω.

Prove that the mapping

$$I(f) : L^2(\Omega) \to L^2(\Omega)$$

given by

$$I(f) = u_f, \text{ where } u_f \text{ is the unique weak solution of } -\Delta u + u = f \text{ in } \Omega,$$

is a compact operator of $L^2(\Omega) \to L^2(\Omega)$.

(ii) Prove that given any $f \in L^2(\Omega)$, there exists a unique $u_f \in H^1(\Omega)$ that is a weak solution of $-\Delta u + u = f$ in Ω.