Qualifying Exam Real Analysis August 2019

Throughout this exam, $\mathfrak{m} = \mathfrak{m}_n$ denotes Lebesgue measure in $\mathbb{R}^n$. The subindex n is often omitted. Also, A^c denotes the complement of the set A.

(1) Let $F : \mathbb{R} \to \mathbb{R}$ be nondecreasing. It is well-known that F is continuous except for countably many jump discontinuities, but one can say more. Indeed, prove that F is differentiable $\mathfrak{m}$-a.e. (Hint: it could be helpful to first consider the case when F is right-continuous, so that you can use "abstract" theorems on measures.)

(2) Let μ be a finite measure on $(X, \mathcal{A})$ and f, and $f_1, f_2, \ldots$ be real-valued $\mathcal{A}$-measurable functions on X. Show that $\{f_n\}$ converges to f in measure if and only if each subsequence of $\{f_n\}$ has a further subsequence that converges to f almost everywhere.

(3) Let $1 < p < \infty$. Suppose that $\{f_n\}_{n \geq 1}$ is a sequence of functions on $[0, 1]$ such that

$$\| f_n \|_{L^p(\mathfrak{m})} \leq 1, \qquad n \geq 1.$$

Let f be an integrable function on $[0, 1]$ such that

$$\lim_{n \to \infty} \int_0^1 f_n h \, d\mathfrak{m} = \int_0^1 f h \, d\mathfrak{m} \ ,$$

for every $h \in L^\infty(\mathfrak{m})$. Prove that $f \in L^p(\mathfrak{m})$.

(4) Let μ and ν be positive measures on $(X, \mathcal{A})$ such that for each positive ε there is a set $A \in \mathcal{A}$ that satisfies $\mu(A) < \varepsilon$ and $\nu(A^c) < \varepsilon$. Show that $\mu \perp \nu$.

(5) If $1 \leq p < \infty$, prove that translation is continuous in the $L^p(\mathbb{R}^n)$ norm, i.e. that if $f \in L^p(\mathbb{R}^n)$, then $\lim_{h \to 0} \| f(\cdot - h) - f(\cdot) \|_{L^p(\mathbb{R}^n)} = 0$. Give a counterexample for this statement if $p = \infty$.