Real Analysis qualification exam: August 2023

Note: all statements require proofs. You can make references to standard theorems from the course; however, you need state the relevant part of the theorem in your own words, unless it's a well known named theorem. For example, "we had a theorem in the class that said that any continuous function on a compact subset of $\mathbb{R}^n$ is uniformly continuous" is a good reference, and "by the uniqueness theorem from the class, f is unique" is not a good reference.

(1) Prove or give a counterexample: if $f_t\colon \mathbb{R} \to \mathbb{R}$ is a Borel measurable function for every $t \in \mathbb{R}$ and $f\colon \mathbb{R} \to (-\infty, +\infty]$ is defined by

$$f(x) = \sup\{f_t(x)\colon t \in \mathbb{R}\},$$

then f is Borel measurable.

(2) Let $(X, \mathcal{S}, \mu)$ be a measure space with finite measure, $f\colon X \to \mathbb{R}$ be a measurable function, and $f_n\colon X \to \mathbb{R}$ be a sequence of measurable functions. Suppose that $f_n \to f$ in measure. Show that $f_n^2 \to f^2$ in measure.

(3) Suppose that $f \in L^p([0,1])$ for some $p > 1$ (with respect to the standard Lebesgue measure). Show that

$$\lim_{t \to 0+} t^{-1+\frac{1}{p}} \int_0^t f(x)\, dx = 0.$$

(4) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $E_n \in \mathcal{S}$ be a sequence of measurable subsets, and

$$A = \{x \in X\colon x \in E_n \text{ for infinitely many values of } n\}.$$

Suppose that $\mu(E_n) \to 0$ as $n \to +\infty$. Does it always imply that $\mu(A) = 0$? Prove it or give a counterexample.

(5) Let $(X, \mathcal{S}, \mu)$ be a measure space with finite measure, and $f\colon X \to [0, +\infty]$ be a measurable function, satisfying

$$\mu(\{x \in X\colon f(x) > \lambda\}) < \frac{1}{(1+\lambda)^2}$$

for all $\lambda > 0$. Show that $f \in L^1(X, \mu)$.