Real Analysis qualification exam: January 2023

Note: all statements require proofs. You can make references to standard theorems from the course; however, you need state the relevant part of the theorem in your own words, unless it's a well known named theorem. For example, "we had a theorem in the class that said that any continuous function on a compact subset of $\mathbb{R}^n$ is uniformly continuous" is a good reference, and "by the uniqueness theorem from the class, f is unique" is not a good reference.

(1) Find three sequences of Borel functions $f_n, g_n, h_n \colon \mathbb{R} \to \mathbb{R}$ such that:
  (a) $f_n \to 0$ in $\mathcal{L}^1(\mathbb{R})$ but does not converge almost everywhere.
  (b) $g_n \to 0$ in measure but converges neither in $\mathcal{L}^1(\mathbb{R})$ nor almost everywhere.
  (c) $h_n \to 0$ almost everywhere on $\mathbb{R}$ but converges neither in $\mathcal{L}^1(\mathbb{R})$ nor in measure.

  In all types of convergence, use the Lebesgue measure.

(2) Let $f \in \mathcal{L}^p([0,3]^2, \mu)$, where μ is the standard Lebesgue measure on $[0,3]^2 \subset \mathbb{R}^2$ and $1 < p < +\infty$. Show that

$$\left| \int_{[0,3]^2} f \, d\mu \right| \leqslant C_p \|f\|_p,$$

where C_p depends on p but not on f. Find the smallest possible value of such C_p.

(3) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(y) - f(x) \geqslant y - x$ for all $x, y \in \mathbb{R}$ with $x < y$. Show that $|f^{-1}(A)| = 0$ for every $A \subset \mathbb{R}$ with $|A| = 0$. Here $|\cdot|$ denotes the Lebesgue outer measure. Hint: note that f is strictly monotone and one-to-one.

(4) Let $(X, \mathcal{S}, \mu)$ be a measure space with $\mu(X) < +\infty$. Let $f \colon X \to [0, +\infty)$ be a measurable function. Show

$$\int_X e^{f(x)} \, dx = \mu(X) + \int_0^{+\infty} \mu(\{x \in X \colon f(x) \geqslant s\}) e^s \, ds.$$

(5) Let $(X, \mathcal{S}, \mu)$ be a measure space with a σ-finite measure. Suppose that $f_n \colon X \to \mathbb{R}$ are measurable functions, and $f_n \to f$ almost everywhere on X. Show that there exist measurable subsets $\{E_k\}_{k \in \mathbb{N}}$ such that f_n converges to f uniformly on E_k, and

$$\mu\left(X \setminus \cup_{n=1}^{\infty} E_n\right) = 0.$$