Note: all statements require proofs. You can make references to standard theorems from the course; however, you must state the relevant part of the theorem in your own words, unless it is a well-known named theorem. For example, "by the monotone convergence theorem," or, "we showed in lecture that the integrals of an increasing sequence of positive functions converge to the integral of their limit," are good references but, "by a convergence theorem the integrals converge," is **not** a good reference.

1. Let μ be a regular Borel measure on $\mathbb{R}$, let $f\colon \mathbb{R} \to \mathbb{C}$ be a continuous function with compact support, and let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence converging to $a \in \mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x - a_n)\ d\mu(x) = \int_{\mathbb{R}} f(x - a)\ d\mu(x).$$

2. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. For $f \in L^1(X, \mu)$, show that

$$\int_0^{+\infty} t\,\mu(\{x \in X\colon |f(x)| \geq t^2\})\ dt = \frac{1}{2}\|f\|_1.$$

3. Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of complex measures on a measurable space $(X, \mathcal{M})$. Show that if

$$\sum_{n=1}^{\infty} |\nu_n|(X) < \infty,$$

then $\nu(E) := \sum_{n=1}^{\infty} \nu_n(E)$ is a complex measure.

4. Let $\mathcal{B}$ and m denote the Borel σ-algebra and Lebesgue measure on $\mathbb{R}^2$, respectively. For $\theta \in \mathbb{R}$ let $R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the map that rotates the plane about the origin by θ in the counterclockwise direction. Show that for all $E \in \mathcal{B}$ one has $R_\theta(E) \in \mathcal{B}$.

5. Show that a function $F\colon \mathbb{R} \to \mathbb{C}$ is Lipschitz continuous with constant M if and only if F is absolutely continuous and $|F'| \leq M$ m-almost everywhere.

   [**Note:** recall that Lipschitz continuous means there exists $M > 0$ so that $|F(x) - F(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$.]