Real Analysis qualification exam

Note: all statements require proofs. You can make references to standard theorems from the course, but you need to specify which exact standard fact you are referring to.

1. Show that the set of real numbers whose decimal expansion has digit 5 appearing infinitely often is a Borel set.

2. Suppose that $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) < +\infty$, and $f_1, f_2, \ldots$ is a sequence of measurable functions $f_j \colon X \to \mathbb{R}$ such that $f_j(x) \to +\infty$ for all $x \in X$. Show the following: for every $\varepsilon > 0$ there exists $E \in \mathcal{S}$ with $\mu(X \setminus E) < \varepsilon$, such that f_j converges to $+\infty$ uniformly on E. The uniform convergence here means: for every $M > 0$ there exists $N > 0$ such that for all $j > N$ and $x \in E$, we have $f_j(x) > M$. Hint: this statement is similar to Egorov's theorem.

3. Let $f \colon \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function. Show the following: for every $\varepsilon > 0$ there is $\delta > 0$ such that for every Borel $E \subset \mathbb{R}$ with $\mu(E) < \delta$ we have $\mu(f(E)) < \varepsilon$.

4. Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space, and $f \colon X \to [0, +\infty)$ be a measurable function. Show:
$$\int_X f \, d\mu = 2 \int_{[0,+\infty]} \lambda \mu\{x \in X \colon \sqrt{f(x)} > \lambda\} \, d\lambda.$$

5. Let $(X, \mathcal{S}, \mu)$ be a measure space with $\mu(X) = 1$. Show that $f \in \mathcal{L}^\infty(X, \mu)$ if and only if $f \in \mathcal{L}^p(X, \mu)$ for all $p \in [1, +\infty)$ and
$$\sup_p \|f\|_p < +\infty.$$