Note: all statements require proofs. You can make references to standard theorems from the course; however, you must state the relevant part of the theorem in your own words, unless it is a well-known named theorem. For example, "by the monotone convergence theorem," or, "we showed in lecture that the integrals of an increasing sequence of positive functions converge to the integral of their limit," are good references but, "by a convergence theorem the integrals converge," is **not** a good reference.

Throughout m will denote the Lebesgue measure on $\mathbb{R}$.

1. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) := \sum_{n=1}^{\infty} 1_{[-n,n]}(t) \frac{t}{n^4}.$$

   Show that $f \in L^1(\mathbb{R}, m)$ and $\int_{\mathbb{R}} f \, dm = 0$.

2. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and for a sequence of $\mathcal{M}$-measurable functions $f_n\colon X \to [0,1]$, $n \in \mathbb{N}$, define a sequence of measures on $(X, \mathcal{M})$ by $d\nu_n := f_n d\mu$ for all $n \in \mathbb{N}$. Show that $f_n \to 1$ in measure with respect to μ if and only if

$$\lim_{n\to\infty} \nu_n(X) = \mu(X).$$

3. Let $f\colon \mathbb{R} \to \mathbb{C}$ be bounded and uniformly continuous, and let $g \in L^1(\mathbb{R}, m)$. For $x \in \mathbb{R}$, define

$$h(x) := \int_{\mathbb{R}} f(x-y)g(y) \, dm(y).$$

   (a) Show that h is a bounded and uniformly continuous function on $\mathbb{R}$.

   (b) Suppose $\lim_{|x|\to\infty} f(x) = 0$. Show that $\lim_{|x|\to\infty} h(x) = 0$.

4. Let μ, ν be equivalent, positive measures on a measurable space $(X, \mathcal{M})$, and let $\frac{d\nu}{d\mu}$ denote the Radon–Nikodym derivative of ν with respect to μ. For positive numbers $0 < a < b$, show that $a \leq \frac{d\nu}{d\mu} \leq b$ almost everywhere if and only if $a\mu(E) \leq \nu(E) \leq b\mu(E)$ for all $E \in \mathcal{M}$.

5. Suppose μ is a positive Borel measure on $\mathbb{R}$ that satisfies $\mu(E) \leq m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}}$. Show that $F\colon \mathbb{R} \to \mathbb{R}$ defined by

$$F(t) := \begin{cases} \mu([0,t)) & \text{if } t \geq 0 \\ -\mu([t,0)) & \text{otherwise} \end{cases},$$

   is absolutely continuous.