Real Analysis qualification exam: August 2025

Note: all statements require proofs. You can make references to standard theorems from the course; however, you need state the relevant part of the theorem in your own words, unless it's a well known named theorem. For example, "we had a theorem in the class that said that any continuous function on a compact subset of $\mathbb{R}^n$ is uniformly continuous" is a good reference, and "by the uniqueness theorem from the class, f is unique" is not a good reference.

(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function, not necessarily measurable. Show that

$$\{x \in \mathbb{R}\colon f \text{ is continuous at } x\}$$

is a Borel subset of $\mathbb{R}$. Hint: one can actually show that it is a countable intersection of open sets.

(2) For each pair of spaces:

(a) $L^2([0,1])$ and $L^3([0,1])$ with the standard Lebesgue measure;
(b) $L^2(\mathbb{R})$ and $L^3(\mathbb{R})$ with the standard Lebesgue measure;
(c) $\ell^2(\mathbb{Z})$ and $\ell^3(\mathbb{Z})$ with the standard counting measure,

is one of them is included in the other? For example, $\ell^2(\mathbb{Z}) \subset \ell^3(\mathbb{Z})$ or $\ell^3(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$? Provide the proof of each corresponding inclusion, or a proof that neither is included in the other. Use the usual conventions about identifying functions coinciding on zero measure sets.

(3) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function. Show that the graph of f has measure zero in $\mathbb{R}^2$.

(4) Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) < +\infty$, and $f_n\colon X \to [0, +\infty)$ be measurable functions. Show that $f_n \to 0$ in measure if and only if

$$\int_X \frac{f_n^2}{1 + f_n^2}\, d\mu \to 0, \quad \text{as } n \to \infty.$$

(5) Let $(X, \mathcal{S}, \mu)$ be a measure space with $\mu(X) < +\infty$, and $f\colon X \to (0, +\infty)$ be a strictly positive measurable function. Show that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, f) > 0$ such that, for every measurable subset $E \subset X$ with $\mu(E) \geqslant \varepsilon$, we have

$$\int_E f(x)\, dx \geqslant \delta.$$

Hint: consider the sets $\{x \in X\colon f(x) \geqslant 1/n\}$.