Topology Qualifying Examination

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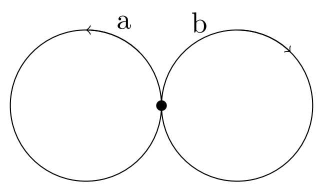
Instructions: Solve four out of the five problems. Justify your claims either by direct arguments or by referring to theorems you know.

Problem 1. State and prove the Brouwer fixed point theorem.

Problem 2. Let Z be a CW complex constructed from the 1-complex shown here:



by attaching three 2-cells via attaching maps that traverse the loops a^5, b^3, and $aba^{-1}b^{-1}$, respectively, and a single 7-cell (attached along any map)

a) Compute $\pi_1(Z)$.

b) Prove that any map $f : Z \to S^1$ is homotopic to a constant map.

Problem 3.

a) Let Y be any topological space. **Using only the definition of singular homology**, prove that $H_0(Y)$ is isomorphic to the free abelian group generated by the path components of Y.

b) Let $+$ denote the subspace of $\mathbb{R}^2$ consisting of the coordinate axes; that is,

$$+ := \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$$

Prove there does **NOT** exist a homeomorphism $g : + \longrightarrow +$ sending the origin $p = (0, 0)$ to the point $q = (1, 0)$. (Hint: Part (a) is there for a reason.)

Problem 4. Find all the connected covering spaces of $\mathbb{RP}^2 \times \mathbb{RP}^2$. Justify your answer carefully. Which of these covering spaces are normal? Explain.

Problem 5. Let X be the space consisting of a genus two surface Σ and a 2-disk D, meeting along a circle as shown in the figure on the following page. Calculate the homology groups $H_i(X)$ for all i.

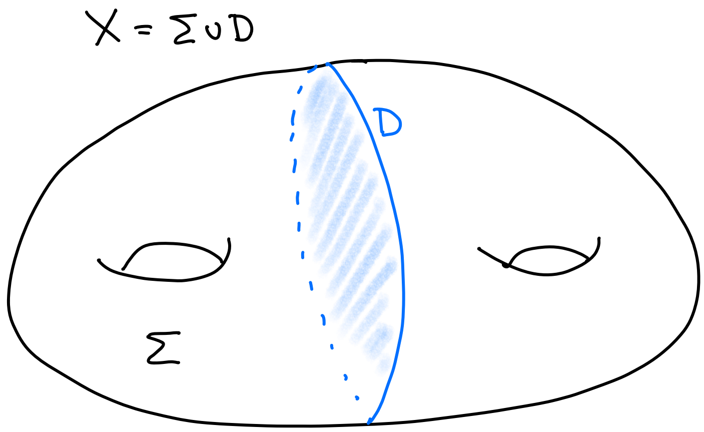


FIGURE 1. The space X in problem 6.