TOPOLOGY QUALIFYING EXAM– FALL 2023

Instructions: Solve any FOUR out of the problems. If you work on all problems, your top four problem scores will count towards your exam grade.

You have two hours to work on the exam. **NOTE:** In your solutions you should not use theorems that are outside the scope of the material covered in this course.
Show your work; all answers must be justified appropriately.

Problem 1 Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.

Problem 2. Prove that there is no open cover $\{U, V\}$ of $\mathbb{RP}^2$ where U, V are two open, connected sets, and U, V are both contractible, and $U \cap V$ is connected.

Problem 3. Let X be a topological space and let $Y = (X \times [0,1])/\sim$ where $\sim$ is defined by $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$ for all $x, x' \in X$. Compute the homology groups of Y in terms of those of X.

Problem 4. (a) Explain how to construct a CW complex X with fundamental group $\pi_1(X) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/5\mathbb{Z}$.
(b) Does there exist a space X with $\pi_1(X)$ as in part (a), and $H_2(X) = \mathbb{Z}$? If so, construct, otherwise, prove there does not exist such X.

Problem 5.

(1) Show that every map $f : S^2 \to S^1 \times S^1$ is nullhomotopic.
(2) Give an example of a (connected) non-normal cover of $S^1 \vee S^1$. What is the minimal number of sheets of a connected non-normal cover of $S^1 \vee S^1$?