TOPOLOGY QUALIFYING EXAM – MAY 2021

Instructions: Solve any FOUR out of the five problems. If you work on all five problems, your top four problem scores will count towards your exam score.

NOTE: The only resources you are allowed to use on this exam are the textbook for the course, any personal notes you have created (including old assignments), and the lecture notes for the course on D2L. You cannot use internet resources during this exam. In your solutions you should not use theorems from the book that are outside the scope of the material covered in this course.

All work must be shown, and all answers justified appropriately.

1. (a) Let X be a space that is path-connected, locally path-connected, and $\pi_1(X)$ is a finite group. Show that for any such X, every map $g : X \to S^1$ is nullhomotopic.

(b) True or False: For a space Y and map $f : Y \to Y$, if the induced map $f_* : \pi_1(Y) \to \pi_1(Y)$ is the zero map, then the map f is nullhomotopic. Justify your answer with a proof or a counterexample.

2. (a) Explain how to construct a CW-complex X with homology groups as follows:

$H_0(X) = \mathbb{Z}$
$H_1(X) = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$H_2(X) = \mathbb{Z}/2\mathbb{Z}$
$H_n(X) = 0$ for all $n \geq 3$.

(b) For the space X you constructed in part (a), compute the cohomology groups, $H^i(X; \mathbb{Z}/2\mathbb{Z})$.

3. Let F_n denote the free group on n generators.

(a) Use covering space theory to prove that all index two subgroups of F_2 are isomorphic to F_3.

(b) Give an example of an explicit index 2 subgroup of F_2 that is isomorphic to F_3.

4. Prove or find a counterexample for each of the following assertions:

(a) If a space Y is the union of two subspaces A and B, then the fundamental group $\pi_1(Y)$ is a quotient of the free product $\pi_1(A) * \pi_1(B)$.

(b) For topological spaces X, Y, we have $H_i(X \times Y) \cong H_i(X) \times H_i(Y)$ for all $i > 0$.

(c) There is no retraction $r : X \to A$ from the Möbius band X to its boundary circle A.

5. Let Z be the subspace of $\mathbb{R}^3$ consisting of the unit 2-sphere and two diameters (see picture below). Compute the homology groups $H_i(Z)$.