Topology Qualifying Examination

August 2025

Instructions: Solve **four** out of the **five** problems below. Even if you attempt all the problems, indicate which problems you want graded. You must justify your claims either by direct arguments or by referring to theorems you know.

Important Note: You may NOT use the textbook or any notes during the exam.

Problem 1. Let $\mathbb{R}P^n$ be real projective n-space.

(a) Using the CW structure with one cell in each dimension $0, 1, \ldots, n$, compute $H_i(\mathbb{R}P^n; \mathbb{Z})$ for all i. State precisely for which n the top homology group is $\mathbb{Z}$ and for which it is 0.

(b) Compute the mod-2 cohomology groups $H^i(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$.

(c) Describe the structure of the cohomology ring $H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$. You don't need to give a proof of your answer.

(d) Let $m < n$. Prove that there is no retraction $r : \mathbb{R}P^n \to \mathbb{R}P^m$. (Hint: Use part (c).)

Problem 2. Let $X = S^1 \vee S^1$ be the wedge of two circles with wedge point x_0. Let the loops generating the two circles be a and b.

(a) Compute $\pi_1(X, x_0)$ in in terms of a, b.

(b) Classify (up to equivalence of coverings) all connected 2-sheeted covering spaces $p : Y \to X$. Give explicit labeled graph models for the total spaces Y, indicate which coverings are normal, and for each normal cover compute the deck transformation group.

(c) Draw a picture of a covering space corresponding to the normal subgroup of $\pi_1(X, x_0)$ generated by a, b and $(ab)^4$, and prove this covering space is indeed the correct one.

Problem 3

(a) Let D^2 denote the 2-dimensional unit disc. Is there a continuous function $f : D^2 \longrightarrow D^2$ without fixed points?

(b) Is there a connected, compact 4-dimensional manifold without boundary such that: X is simply connected and $H_3(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$?

(c) Suppose that Y is an n-dimensional embedded submanifold of a simply connected $n+2$-dimensional manifold X. Is the complement $X \setminus Y$ simply connected?

Problem 4. Let $E \subset S^2$ be the equator of the 2-sphere and $f : E \longrightarrow S^1$ be an n-sheeted *covering map* of E onto S^1 (where $n > 0$). Form the quotient space X_n from the disjoint union $S^2 \coprod S^1$ by identifying each $x \in E \subset S^2$ with $f(x) \in S^1$.

(a) Calculate the fundamental group of X_n.

(b) The quotient map in part (a) gives a map $q : S^2 \longrightarrow X_n$. Calculate $H_2(X_n)$ and describe the map $q_* : H_2(S^2) \longrightarrow H_2(X_n)$, induced by q.

(c) Let $\widetilde{X}_n$ denote the universal cover of X_n. Show that for every n, there is a continuous map $S^2 \longrightarrow \widetilde{X}_n$ not homotopic to a constant map.

Problem 5.

(a) Let $\mathbf{R}P^2$ denote the real projective plane and let $M = S^1 \times S^1 \times S^1$. Prove that any continuous map
$$f : \mathbf{R}P^2 \times \mathbf{R}P^2 \longrightarrow M$$
is null-homotopic.

(b) Show that if M is a compact contractible n-manifold for $n \geq 1$, then ∂M is a homology $(n-1)$-sphere. That is, we have $H_i(\partial M) = H_i(S^{n-1})$, for all i.