Topology Qualifying Examination

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Instructions: Solve **four** out of the **five** problems. Even if you attempt more than four problems, indicate which problems you want graded. You must justify your claims either by direct arguments or by referring to theorems you know.

Problem 1. For $n \geq 1$, let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. Prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ for any $m \neq n$.

Problem 2. Let F_2 denote the free group of rank two. Use covering spaces to show that any index 2 subgroup of F_2 is a free group on three generators.

Problem 3. Suppose that X is a finite CW-complex and let Y be an an n-sheeted covering space of X. Prove that Euler characteristics have the following relation: $\chi(Y) = n \cdot \chi(X)$. *Hint:* Show that a CW-structure on X lifts to one on Y.

Problem 4. Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$ be the wedge of two real projective planes.

(a) Is $\pi_1(X)$ isomorphic to $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2)$? Justify your answer.

(b) Are the homology groups $H_i(X)$ isomorphic to $H_i(\mathbb{R}P^2 \times \mathbb{R}P^2)$, for all i? Justify your answer.

(c) Is there a continuous map $f : \mathbb{R}P^2 \times \mathbb{R}P^2 \longrightarrow S^1 \times S^1$ that is not homotopic to a constant map? Justify your answer.

Problem 5. Let $E \subset S^2$ be the equator of the 2-sphere and $f : E \longrightarrow S^1$ be an n-sheeted *covering map* of E onto S^1 (where $n > 0$). Form the quotient space X_n from the disjoint union $S^2 \coprod S^1$ by identifying each $x \in E \subset S^2$ with $f(x) \in S^1$.

(a) Calculate the fundamental group of X_n.

(b) The quotient map in part (a) gives a map $q : S^2 \longrightarrow X_n$. Calculate $H_2(X_n)$ and describe the map $q_* : H_2(S^2) \longrightarrow H_2(X_n)$, induced by q.

(c) Let $\widetilde{X}_n$ denote the universal cover of X_n. Show that for every n, there is a continuous map $S^2 \longrightarrow \widetilde{X}_n$ not homotopic to a constant map.