The traditional **Herzog competition** of year 2022 took place on November 19. Seventeen MSU undergraduate students had three hours to solve six problems. The winners are

First prize:	Erik Brodsky
Second prize:	Duong Tran
Third prize:	Evan Bell
	William Carvalho
	Joseph Noonan
	John Wong
Honorable	Aamierul Harith Helmi

Here is the list of problems

Question 1. Find all natural a and b such that neither of these two numbers is a multiple of the other number and

$$\frac{1}{a} + \frac{1}{b} = \frac{5}{2022}.$$

Question 2. Fibonacci numbers F_n are defined as $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 3$. Let $S(r)$ denote the number of possibilities to write F_r as a sum of Fibonacci numbers with distinct numbers (say, $F_r = F_r = F_{r-1} + F_{r-2} = \cdots$). Find $S(2022)$.

Question 3. Let $f : \mathbb{R} \to \mathbb{R}$ be an everywhere differentiable function defined for all real x and let

$$f(x) = f(x/2) + \frac{x}{2} f'(x)$$

for any $x \in \mathbb{R}$. Prove that $f(x)$ is a linear function, that is, $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Question 4. Define functions $Q_n(x)$ recurrently, $Q_0(x) = 1$, $Q_1(x) = x$ and

$$Q_{n+1}(x) = \frac{Q_n^2(x) + (-1)^{n+1}}{Q_{n-1}(x)}.$$

Prove that $Q_n(x)$ is a polynomial for all natural n.

Question 5. Let a triangle-shaped solid angle be bounded by three edges parallel to three unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ in $\mathbb{R}^3$ and by three sides lying in the planes spanned by $(\mathbf{i}_1, \mathbf{i}_2)$, $(\mathbf{i}_1, \mathbf{i}_3)$ and $(\mathbf{i}_2, \mathbf{i}_3)$. Define the "mean line" of such a solid angle as the line passing through the summit of the angle in the direction $\mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$.

Prove that the mean lines of all four solid angles of a tetrahedron pass through the same point inside the tetrahedron if and only if all pairwise products of lengths of opposite sides of the tetrahedron are equal to the same number.

Question 6. Find all positive integer solutions x, y, z to the equation

$$5^x + 12^y = z^2.$$