Herzog competition. MSU Department of Mathematics, November 9 2024

Please, write on a separate paper provided. You may take this list of problems with you.

Question 1. Let A and B be two different matrices of size $n \times n$ with real entries. Given that $A^4 = B^4$ and $A^3B = B^3A$ can $A^3 + B^3$ be invertible?

Question 2. Find all real polynomials of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that $p(r_i) = 0$, $i = 1, \ldots n$ and

$$p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0, \quad i = 1, \ldots n - 1,$$

where $p'(x)$ denotes the derivative of $p(x)$.

Question 3. Show that the equation $y^2 = x^5 - 4$ has no integer solutions.

Question 4. Let $f : [a, b] \to \mathbb{R}$ be a continuous function that is differentiable on (a, b). Assume that f is nowhere zero on (a, b). Prove that there exists $\theta \in (a, b)$ such that

$$\frac{f'(\theta)}{f(\theta)} = \frac{1}{a - \theta} + \frac{1}{b - \theta}.$$

Question 5. We have 2024 distinct points on a unit circle so that the product of distances from any point on the circle to all the given points does not exceed 2. Prove that the points are vertices of a regular 2024-gon.

Question 6. Let us have a board of size $1 \times N$ (that is, the board composed out of N consecutive squares). Domino tiles (of size 1×2) are placed on this board without overlapping such that each tile occupies two neighbor squares. A configuration of tiles is called **admissible** if no more tiles can be placed on the board without overlapping. Let T_N be the sum over all admissible configurations p taken with weights $f(p) := 6^{k(p)}$, where $k(p)$ is the number of remaining empty squares in p. Find $\lim_{N\to\infty} 2^{-N} T_N$.