THE FIFTH ANNUAL HERZOG PRIZE EXAMINATION

November 12, 1977

Problem 1: (D. Moran) Let M be an $n \times n$ matrix of integers whose inverse is also a matrix of integers. Prove that the number of odd entries in M is at least n and at most $n^2 - n + 1$.

Problem 2: (A.M.M.E 1297) Having chosen $0 < a_1, b_1 < 1$ define recursively

$$a_{n+1} = a_1(1 - a_n - b_n) + a_n$$

and

$$b_{n+1} = b_1(1 - a_n - b_n) + b_n.$$

Prove that $\lim_n a_n$ and $\lim_n b_n$ both exist and evaluate these limits.

Problem 3: (L. Kelly) Consider the binary homogeneous quadratic form $x^2 + bxy + y^2$. Suppose it is known that this form produces perfect integral squares for all positive integral choices of x and y. Prove that b must be ± 2.

Problem 4: (L. Kelly) A solid sphere rolls on a plane π always touching a fixed line L. Find the locus of its center.

Problem 5: (L. Kelly) Show that if all the distances between pairs of points of a seven point subset of the unit disc are at least 1, then the points must be the vertices of a regular inscribed hexagon and the center of the circle.

Problem 6: (A.M.M.E 1342) If $x, y > 0$, prove that

$$x^y + y^x > 1.$$