THE TWELFTH ANNUAL HERZOG PRIZE EXAMINATION

November 3, 1984

Problem 1: (L.M. Kelly) Assume that the points of the plane are each colored red, white, or blue.

Prove that there are two points 1 unit distance apart of the same color.

Problem 2: (J.S. Frame) Evaluate

$$\int_0^{\pi/2} \ln(2\sin 2\theta)\,d\theta \quad .$$

Problem 3: (Alan Parks) Let $a_1 < a_2 < \ldots < a_n$ be a list of positive integers such that for every $i \neq j$, at least one of

$$a_i - a_j,\ a_j - a_i, \text{ or } a_i + a_j$$

is again among the list. An example is $a_i = ik$ for a constant k. Show that if the list is not of this form, then $n = 3$.

Problem 4: (Kevin McCurley) Show that $f(n) = n^6 + 671$ is composite for $n = -83, -82, -81, \ldots, 0, 1, 2, \ldots, 82, 83$.

Problem 5: (L.M. Kelly) If x and y are positive numbers and m and n positive integers, prove

$$x^m y^n/(x+y)^{m+n} \leq m^m n^n/(m+n)^{m+n} \ .$$

Problem 6: (Chris Bishop) Given a disc of radius R and n discs of radius $r < R$ inside, prove that there exists a line segment intersecting at least $[nr/R]$ of the smaller discs. ($[x]$ denotes the greatest integer less than or equal to x).