THE NINETEENTH HERZOG PRIZE EXAMINATION

November 16, 1991

1. The integer 1991 is a palindromic number. (It reads the same backward and forward). Prove that every palindromic number with an even number of digits (written as usual in base 10) is divisible by 11.

2. In the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ defined by $a_0 = a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$, express $a_{n-1}a_{n+1}$ in terms of a_n and prove that the resulting expression is correct.

3. Let f be a continuous real-valued function defined on the reals, and suppose that f has no fixed points (i.e. that $f(x) \neq x$ for every x). Prove that the iterated functions $f^2 = f \circ f$, $f^3 = f \circ f \circ f, \ldots$ also have no fixed points.

4. Let 3 points be contained in a parallelogram of area 1. Prove that the triangle determined by these points has area not exceeding $\frac{1}{2}$.

5. Let f be a differentiable real-valued function on the reals. Suppose that $f(0) = 0$, and that its first derivative f' is itself differentiable and monotone increasing. Prove that the function g defined by $g(x) = f(x)/x$ is monotone increasing on $(0, \infty)$.

6. Let $P(x)$ be a polynomial of degree greater than 2, all of whose zeros $r_1 < r_2 < \ldots < r_n$ are real and distinct. Let c be the critical value in the interval (r_1, r_2) whose existence is guaranteed by Rolle's theorem. Prove that c is nearer to r_1 than to r_2. (Assume that $P(x)$ has leading coefficient 1 for simplicity).