THE TWENTY-THIRD HERZOG PRIZE EXAMINATION

November 11, 1995

1. Find two non-trivial factors of $x^{10} + x^5 + 1$.

2. Prove or disprove: The binomial coefficient $\binom{2n}{n}$ is always an even number for $n \geq 1$.

3. Consider the plane curve defined by the parametric equations

$$\begin{cases} x = \sin \pi t \\ y = \cos t \end{cases}, \quad t \in [0, \infty).$$

The points on this curve all lie in the square $\{(x, y) : -1 \leq x \leq 1, \ -1 \leq y \leq 1\}$. Do <u>all</u> of the points in this square lie on this curve, or are there some that are not on the curve?

4. Prove that $2^{mn} - 1$ is divisible by $2^m - 1$ for every pair of positive integers m and n.

5. A wide-beam spotlight can cast its light on all points lying within and on the boundary of a $90°$ angle. Four such lamps are placed at arbitrary points in the plane. Prove that it is always possible to direct the beams so that the entire plane will be illuminated.

6. Find all continuous real-valued functions that satisfy $f(x)f(y) = f(x - y)$ for every pair of real numbers x and y.