Exploring Stability in Geometric and Functional Inequalities: OT and beyond

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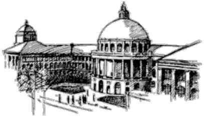

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Euler and the Calculus of Variations

Leonhard Euler (1707-1783)

1744: *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes sive Solutio Problematis Isoperimetrici Latissimo Sensu Accepti*

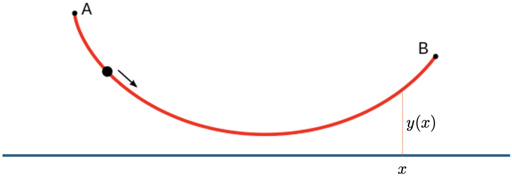
(A method for finding curved lines enjoying properties of maximum or minimum, or solution of isoperimetric problems in the broadest accepted sense)

Euler and the Calculus of Variations

Question: given a function Z, find a minimum/maximum of

$$y \mapsto \int Z(x, y(x), y'(x))\, dx.$$

E.g. Find a plane curve between two points along which a particle descends in the shortest time under the influence of gravity



Remark: In solving the brachystochrone problem, Euler used a simple instance of the Lagrange multiplier method.

Euler and the Calculus of Variations

Giuseppe Luigi Lagrangia (1736-1813)

In 1755, the 19-year-old Lagrange wrote Euler a brief letter to which he attached a mathematical appendix with the revolutionary technique of *variations*.

Euler and the Calculus of Variations

Euler said:

"*Even though the author of this had meditated a long time and revealed to friends his desire, yet the glory of first discovery was reserved to the very penetrating geometer of Turin La Grange, who having used analysis alone, has clearly attained the very same solution which author had deduced by geometrical considerations.*"

Euler dropped his method, embraced that of Lagrange, and renamed the subject *Calculus of Variations.*

The Calculus of Variations

Fundamental problem in the Calculus of Variations: find minima/maxima of functionals.

Two (by-now) classical examples:

- **Isoperimetric inequalities**
- **Brunn-Minkowski inequality**

The Calculus of Variations

Basic question:

Find/characterize the minimizers.

Next natural question:

Are minimizers stable?

That is, if a function/set almost attains equality, is it close to one of the minimizers?

Overview of the talk

1. Stability for isoperimetric inequalities

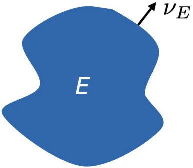
2. Stability for the Brunn-Minkowski inequality

Part 1: Isoperimetric inequalities

Isoperimetric inequalities and stability

Classical isoperimetric inequality: For any bounded smooth set $E \subset \mathbb{R}^n$, the perimeter $P(E)$ controls the volume $|E|$

$$P(E) \geq n|B_1|^{1/n}|E|^{(n-1)/n}$$



Moreover, equality holds if and only if E is a ball.

Stability question: If E is "almost a minimizer", does this imply that E is close to a ball, if possible in some quantitative way?

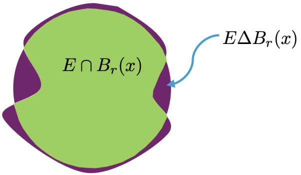
Isoperimetric deficit of E:

$$\delta(E) := \frac{P(E)}{n|B_1|^{1/n}|E|^{(n-1)/n}} - 1\,.$$

- $\delta(E) \geq 0$
- $\delta(E) = 0 \quad \Leftrightarrow \quad E$ is a ball

Asymmetry index of E:

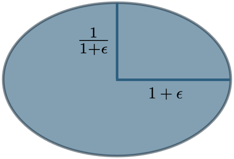
$$A(E) := \inf_x \left\{ \frac{|E\Delta(B_r(x))|}{|E|} : |B_r| = |E| \right\}$$



Question: can we find positive constants $C = C(n)$ and $\alpha = \alpha(n)$ such that

$$A(E) \leq C\,\delta(E)^{\alpha}\,?$$

Remark: by testing the above inequality on a sequence of ellipsoids converging to B_1, we get $\alpha \leq 1/2$.



$$A(E) \sim \epsilon$$

$$\delta(E) \sim \epsilon^2$$

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Remark: by testing the above inequality on a sequence of ellipsoids converging to B_1, we get $\alpha \leq 1/2$.

This is actually the sharp result:

Theorem (Fusco-Maggi-Pratelli 2008, F.-Maggi-Pratelli 2010, Cicalese-Leonardi 2012)

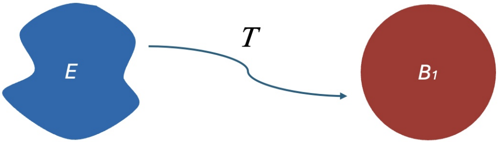
The stability result holds with $\alpha = 1/2$.

Knothe-Gromov's proof of the isoperimetric inequality

Given E smooth and bounded, consider the probability densities

$$f(x) := \frac{\chi_E(x)}{|E|}, \qquad g(y) := \frac{\chi_{B_1}(y)}{|B_1|}.$$

By Optimal Transportation Theory, there exists $\varphi : \mathbb{R}^n \to \mathbb{R}$ convex such that $T := \nabla\varphi$ sends f onto g.



Properties of T:

1. $|T| \leq 1$ in E (since $T(E) \subset B_1$)
2. $\det(DT) = |B_1|/|E|$ (since $T_\# f = g$)
3. $\operatorname{div} T \geq n\big(\det(DT)\big)^{1/n}$ (wait for the next slide).

Then:

$$P(E) = \int_{\partial E} 1 \overset{(1)}{\geq} \int_{\partial E} |T| \geq \int_{\partial E} T \cdot \nu_E$$

$$= \int_E \operatorname{div} T \overset{(3)}{\geq} n \int_E (\det(DT))^{1/n} \overset{(2)}{=} n|B_1|^{1/n}|E|^{(n-1)/n}.$$

Proof of (3)

Since $T = \nabla\varphi$ with φ convex, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $D^2\varphi$ are non-negative.

Hence:

$$\operatorname{div} T = \Delta\varphi = n\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i\right) \geq n\left(\prod_{i=1}^{n}\lambda_i\right)^{1/n} = n\big(\det(DT)\big)^{1/n},$$

where we used the arithmetic-geometric inequality.

Strengths of this proof:

• It works also for more general perimeter-type functionals used to model the surface energy of crystals.

• It is very robust.

In particular, by carefully making “quantitative” each inequality one can prove the desired stability result (F.-Maggi-Pratelli).

Part 2: The Brunn-Minkowski inequality

Semisum of sets

Given $t \in (0,1)$, and $A \subset \mathbb{R}^n$ a Borel set with $|A| > 0$, define

$$tA + (1-t)A := \{ta + (1-t)a' \,:\, a, a' \in A\}.$$

Note that $tA + (1-t)A \supset A$, therefore

$$|tA + (1-t)A| \geq |A|.$$

What about the equality case?

Equality holds iff A is "convex".

More precisely, equality holds iff

$$|\mathrm{co}(A) \setminus A| = 0,$$

where $\mathrm{co}(A)$ denote the convex hull of A.

The Brunn-Minkowski inequality

This is a particular case of a more general inequality: the **Brunn-Minkowski** inequality.

Given $A, B \subset \mathbb{R}^n$ Borel with $|A|, |B| > 0$, define

$$tA + (1-t)B := \{ta + (1-t)b \,:\, a \in A,\, b \in B\}.$$

Then

$$|tA + (1-t)B|^{1/n} \geq t|A|^{1/n} + (1-t)|B|^{1/n}.$$

Equality holds iff A and B are "homothetic convex sets":

there exist $\alpha, \beta > 0$, $v, w \in \mathbb{R}^n$, and $\mathcal{K} \subset \mathbb{R}^n$ convex, such that

$$A \subset \alpha\mathcal{K} + v, \qquad |(\alpha\mathcal{K} + v) \setminus A| = 0,$$

$$B \subset \beta\mathcal{K} + w, \qquad |(\beta\mathcal{K} + w) \setminus B| = 0.$$

The **BM** inequality has applications in:

- convex geometry;

- analysis;

- statistics,

- information theory;

- etc.

From the analytic side, the following chain of implications holds:

$$\mathbf{BM} \quad \Rightarrow \quad \mathbf{Isop.\,ineq.} \quad \Rightarrow$$

$$\Rightarrow \quad \mathbf{Sobolev} \quad \Rightarrow \quad \mathbf{Gagliardo-Nirenberg.}$$

The stability question

Up to rescaling, we can assume $|A| = |B| = 1$.

Also, up to exchanging A and B, we can assume $t \in (0, 1/2]$.

Then BM tells us

$$|tA + (1-t)B| \geq 1$$

Assume that

$$|tA + (1-t)B| - 1 =: \delta_t(A, B) \ll 1$$

Question 1: Is it true that both A and B are almost convex, and that actually they are close to the same convex set?

Question 2: And is it possible to have an explicit information about the dependence on the parameter δ_t?

This stability question has two statements in it:

(Convexity): The error in **BM** controls how far A and B are from their convex hulls.

(Homothety): The error in **BM** controls the difference between the shapes of A and B.

We proceed by steps:

• The **(Homothety)** issue: *assume* that A and B are convex and prove that A and B have almost the same shape.

• The **(Convexity)** issue: *assume* $A = B$ and prove that A is close to its convex hull $\mathrm{co}(A)$.

• Prove the general case.

The homothety issue

Let A, B be bounded convex set with $|A| = |B| = 1$, and define a distance between A and B:

$$d(A,B) := \min_{x\in\mathbb{R}^n} |B\Delta(x+A)|.$$

Theorem (F.-Maggi-Pratelli, 2009)

Let $t \in (0, 1/2]$. Then

$$d(A,B) \leq C_n\, t^{-1/2} \delta_t(A,B)^{1/2}.$$

Remark: the dependence $t^{-1/2}$ and the exponent $1/2$ are optimal. Also, C_n is explicit. The proof is again via optimal transport (McCann 1995 - FMP 2009).

The convexity issue

The proof of quantitative stability for **BM** via optimal transport works *only* if both A and B are convex, and a completely new strategy is needed to address the **(Convexity)** issue.

The case $n = 1$, $t = 1/2$. Let $|A| = 1$, and define

$$\delta_{1/2}(A) := \left|\tfrac{1}{2}A + \tfrac{1}{2}A\right| - 1.$$

Question: does $\delta_{1/2}(A)$ control $|\mathrm{co}(A) \setminus A|$?

Remark: in general, $\delta_{1/2}(A)$ cannot control $|\mathrm{co}(A) \setminus A|$.

A $\qquad$ $|A| = 1$

$\frac{1}{2}A + \frac{1}{2}A$ $\qquad$ $\delta_{1/2}(A) = 1/2$

$\mathrm{co}(A)$ $\qquad$ $|\mathrm{co}(A)| \gg 1$

By Freiman, this is the only thing that can go wrong:

Theorem (Freiman, 1959)

Let $A \subset \mathbb{R}$ *with* $|A| = 1$*, and assume* $\delta_{1/2}(A) < 1/2$*.*

Then

$$|\mathrm{co}(A) \setminus A| \leq 2\,\delta_{1/2}(A).$$

Remark: Freiman's Theorem is about the structure of additive subsets of $\mathbb{Z}$, and it provides a sharp stability result in $1D$ also for $t \neq 1/2$ and $A \neq B$.

The case $n > 1$.

Let $A \subset \mathbb{R}^n$ with $|A| = 1$, $t \in (0, 1/2]$, and define

$$\delta_t(A) := |tA + (1-t)A| - 1.$$

Theorem (Christ, 2012)

If $\delta_t(A) \to 0$ *then*

$$|\mathrm{co}(A) \setminus A| \to 0.$$

Theorem (F.-Jerison, 2013-2019; Van Hintum-Spink-Tiba, 2022)

Let $t \in (0, 1/2]$*. There are computable constants* $C_n, \delta_{t,n} > 0$ *such that the following holds:*

If $\delta_t(A) \leq \delta_{n,t}$*, then*

$$|\mathrm{co}(A) \setminus A| \leq C_n t^{-1} \delta_t(A).$$

Remark: the dependence t^{-1} and the exponent 1 are optimal.

The general case

Theorem (F.-Jerison, 2014; F.-van Hintum-Tiba, 2024)

Let $t \in (0, 1/2]$. There are computable constants $C_n, \delta_{n,t} > 0$ such that the following holds:
If $\delta_t(A, B) \leq \delta_{n,t}$ then there exists a convex set $\mathcal{K} \subset \mathbb{R}^n$ such that, up to a translation,

$$A, B \subset \mathcal{K} \qquad \text{and} \qquad |\mathcal{K} \setminus A| + |\mathcal{K} \setminus B| \leq C_n t^{-1/2}\, \delta_t(A, B)^{1/2}.$$

Remark: In FvHT 2024, we also prove that

$$|\mathrm{co}(A) \setminus A| + |\mathrm{co}(B) \setminus B| \leq C_{n,t}\delta_t(A, B).$$

Thanks for your attention!