# Contents

## 1 Functions and Limits
- **Forest for the Trees**
- 1.4 The Tangent and Velocity Problems
- 1.5 The Limit of a Function
- 1.6 Calculating Limits Using the Limit Laws
- 1.7 The Precise Definition of a Limit
- 1.8 Continuity

## 2 Derivatives
- **Forest for the Trees**
- 2.1 Derivatives and Rates of Change
- 2.2 The Derivative as a Function
- 2.3 Differentiation Formulas
- 2.4 Derivatives of Trigonometric Functions
- 2.5 The Chain Rule
- 2.6 Implicit Differentiation
- 2.7 Rates of Change in the Natural and Social Sciences
- 2.8 Related Rates
- 2.9 Linear Approximations and Differentials

## 3 Applications of Differentiation
- **Forest for the Trees**
- 3.1 Maximum and Minimum Values
- 3.2 The Mean Value Theorem
- 3.3 Derivatives and Graphs
- 3.4 Limits at Infinity; Horizontal Asymptotes
- 3.5 Curve Sketching
- 3.7 Optimization Problems
- 3.8 Newton's Method
- 3.9 Antiderivatives
4 Integrals 101
Forest for the Trees ............................................................... 102
4.1 Areas and Distances ............................................................ 103
Appendix E - Sigma Notation ..................................................... 106
4.2 The Definite Integral ............................................................. 109
4.3 The Fundamental Theorem of Calculus ................................. 112
4.4 Indefinite Integrals and Net Change ....................................... 116
4.5 The Substitution Rule ........................................................... 119

5 Applications of Integration 123
5.1 Area Between Curves ........................................................... 124
5.5 Average Value of a Function .................................................. 127
Chapter 1

Functions and Limits
CHAPTER 1. FUNCTIONS AND LIMITS

Forest for the Trees

The main topics for MTH 132 are slope at a point of various functions and area under a curve (by this we mean area between a function and the $x$-axis).

A problem we can solve:
Find the slope of $f(x) = \frac{x}{2} + 1$ at the point $x = 1$.

A problem we can’t solve:
Find the slope of $f(x) = \frac{x^2}{2} + 1$ at the point $x = 1$.

Another problem we can solve:
Find the area under $f(x) = \frac{x}{2} + 1$ between $x = 0$ and $x = 2$.

Another problem we can’t solve:
Find the area under $f(x) = \frac{x^2}{2} + 1$ between $x = 0$ and $x = 2$.

So for linear functions we know how to solve the key questions of calculus. However as we soon as we move to the next easiest function, quadratics, we start to have issues.

In order to solve these problems we need to develop a theory of limits. In 1.4 we will get a taste for using limits to compute slope at a point. We will spend the rest of Chapter 1 further developing the theory of limits.
### 1.4 The Tangent and Velocity Problems

<table>
<thead>
<tr>
<th>Secant Line</th>
<th>Tangent Line</th>
</tr>
</thead>
</table>

#### Definition(s) 1.4.1

The **average rate of change of** \( y = f(x) \) between \( P(x_1, f(x_1)) \) and \( Q(x_2, f(x_2)) \) is given by

\[
A.R.o.C. = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

Graphically this looks like:

#### Definition(s) 1.4.2

If \( t \) has units of ______ and \( f(t) \) describes the ____________ of an object and has units of distance then the ________________ between \( t_1 \) and \( t_2 \) is given by

\[
v_{\text{ave}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}
\]
Remark 1.4.3. Typically the slope of the tangent line (or instantaneous velocity when applicable) is well approximated by the slope of a secant line over a small interval (the smaller the better).

Pictures:

Example 1.4.4. If a rock is thrown upward on the planet Mars with a velocity of $10\text{m/s}$, its height in meters $t$ seconds later is well approximated by $y = 10t - 2t^2$.

(a) Find the average velocity over the time interval $[1, 2]$.

(b) Find the average velocity over the time interval $[0, 1]$.

(c) Using part (a) and (b) approximate the instantaneous velocity at $t = 1$. 


Example 1.4.5. Suppose \( f(x) = 2 \sin \pi x + 3 \cos \pi x \).

(a) Find the slope of the secant line joining \( P(1, f(1)) \) to \( Q\left( \frac{1}{2}, f\left( \frac{1}{2} \right) \right) \).

(b) Find the slope of the secant line joining \( P(1, f(1)) \) to \( R\left( \frac{3}{4}, f\left( \frac{3}{4} \right) \right) \).

(c) Find the slope of the secant line joining \( P(1, f(1)) \) to \( S\left( \frac{5}{6}, f\left( \frac{5}{6} \right) \right) \).

(d) Approximate the slope of the tangent line at \( P \).

(e) Is there a way to get a better approximation? (explain)
1.5 The Limit of a Function

Section Objective(s):

- Have an intuitive idea of the definition of a limit.
- Find the limits (two-sided, left, and right) of the piecewise defined function given algebraically or graphically.
- Calculate infinite limits and detect vertical asymptotes.

**Definition(s) 1.5.1.** Suppose \( f(x) \) is defined when \( x \) is near the \( a \). Then we write:

\[
\lim_{{x \to a}} f(x) = L
\]

if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) to be sufficiently close to \( a \) (on either side of \( a \)) but \( \neq a \).

**Definition(s) 1.5.2 (Left-hand limit).** We write

\[
\lim_{{x \to a^-}} f(x) = L
\]

if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) to be sufficiently close to \( a \) and \( x \) less than \( a \).

**Definition(s) 1.5.3 (Right-hand limit).** We write

\[
\lim_{{x \to a^+}} f(x) = L
\]

if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) to be sufficiently close to \( a \) and \( x \) greater than \( a \).
1.5. THE LIMIT OF A FUNCTION

Theorem 1.5.4.

if and only if

Definition(s) 1.5.5. The line \( x = a \) is called a [ ] of the curve \( y = f(x) \) if at least one of the following statements is true:

\[
\lim_{x \to a^+} f(x) = \infty \\
\lim_{x \to a^-} f(x) = \infty \\
\lim_{x \to a} f(x) = \infty
\]

\[
\lim_{x \to a^+} f(x) = -\infty \\
\lim_{x \to a^-} f(x) = -\infty \\
\lim_{x \to a} f(x) = -\infty
\]

That is \( f \) can be made arbitrarily ________ (or ________ with ________) by taking \( x \) sufficiently close to \( a \), but not equal to \( a \).

Note: Vertical asymptotes most often occur at \( x = a \) when \( f(x) = \frac{g(x)}{h(x)} \) is fully simplified and we have ________ and ________.
Example 1.5.6. Use the given graph of $f$ to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 2^-} f(x)$

(b) $\lim_{x \to 2^+} f(x)$

(c) $\lim_{x \to 2} f(x)$

(d) $\lim_{x \to 4} f(x)$

(e) $f(2)$

(f) $f(4)$

Example 1.5.7. Draw an example of a graph of a function $f(x)$ that satisfies:

\[
\lim_{x \to 4^-} f(x) = 1, \quad \lim_{x \to 4^+} f(x) = \infty, \quad \text{and} \quad f(4) = 2.
\]
Example 1.5.8. Evaluate the following limits for the function \( f(x) = \begin{cases} 
3x - 2 & \text{if } x < 1 \\
0 & \text{if } x = 1 \\
3 - x & \text{if } x > 1 
\end{cases} \)

(a) \( \lim_{x \to 1^-} f(x) \)

(b) \( \lim_{x \to 1^+} f(x) \)

(c) \( \lim_{x \to 1} f(x) \)

(d) \( \lim_{x \to 2} f(x) \)

Example 1.5.9. Find the vertical asymptotes of the function \( y = \frac{x^2 + 2x + 1}{x + x^2} \)
1.6 Calculating Limits Using the Limit Laws

Section Objective(s):
• Understand the eight limit laws and use these to calculate more complicated limits.
• Recall old algebraic techniques of factoring and multiplying by a conjugate.
• Utilize the Squeeze Theorem to determine more limits.

Theorem 1.6.1 (Limit Laws). Suppose that $c$ is a constant and the limits
\[
\lim_{x \to a} f(x) \text{ and } \lim_{x \to a} g(x)
\]
exist. Then:

1. \[
\lim_{x \to a} [f(x) + g(x)] = \]

2. \[
\lim_{x \to a} [cf(x)] = \]

3. \[
\lim_{x \to a} [f(x)g(x)] = \]

4. \[
\lim_{x \to a} \left[ \frac{f(x)}{g(x)} \right] = \quad \text{provided } \lim_{x \to a} g(x) \neq 0
\]

5. \[
\lim_{x \to a} c = \]

6. \[
\lim_{x \to a} x = \]

7. \[
\lim_{x \to a} x^n = \quad \text{when appropriate.}
\]

8. \[
\lim_{x \to a} [f(x)^n] = \quad \text{when appropriate.}
\]

Example 1.6.2. Evaluate the limit \[
\lim_{x \to 0} \frac{2x^2 - x}{3 + x}
\]
Theorem 1.6.3 (Direct Substitution Property). If $f$ is a polynomial or a rational function and $a$ is in the domain of $f$, then

Example 1.6.4. Evaluate the limit, if it exists $\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 1}$

Example 1.6.5. Evaluate the limit, if it exists $\lim_{h \to 0} \frac{\sqrt{9 + h} - 3}{h}$

Theorem 1.6.6 (Cancellation). If $f(x) = g(x)$ when $x \neq a$, then

provided the limit exists.
Theorem 1.6.7. If \( f(x) \leq g(x) \) when \( x \) is near \( a \) (except possibly at \( a \)) and the limits of \( f \) and \( g \) both exist as \( x \) approaches \( a \), then

\[ \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \]

Theorem 1.6.8 (Squeeze Theorem). If \( f(x) \leq g(x) \leq h(x) \) when \( x \) is near \( a \) (except possibly at \( a \)) and

\[ \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \]

then

Example 1.6.9. Use the Squeeze Theorem to find \( \lim_{x \to 0} \left[ x^2 \cos \left( \frac{2}{x} \right) \right] \)
1.7 The Precise Definition of a Limit

Section Objective(s):

- Review absolute value inequalities and properties of absolute values.
- Explain the precise definition of a limit.
- Practice using the precise definition of the limit to formally calculate two-sided limits

Theorem 1.7.1 (Old algebra).

Theorem 1.7.2 (Old algebra). If \( k > 0 \) then

\[
\text{if and only if}
\]

Which has the equivalent geometric statement:

\[
\text{___ is less than ___________ away from ___}.
\]

Example 1.7.3. Write a distance statement for the inequality \( |4x - 5| < 6 \)

Definition(s) 1.7.4. Let \( f \) be a function defined on some open interval that contains the number \( a \), except possibly at \( a \) itself. Then we say that the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \), and we write

if for every number \( \varepsilon > 0 \) there is a number \( \delta > 0 \) such that
Intuitively that means: You can always find a small enough ________ for $x$ that ________ and is ________ at $a$ to keep $f(x)$ within distance $\varepsilon$ close to $L$.

**Example 1.7.5.** Prove that $\lim_{x \to 1}[3x - 1] = 2$ using the formal definition of a limit.

**Remark 1.7.6** (Strategy for solving problems). Find a ___ based on ___ solving the absolute value inequality. __________________________ (NOT OPTIONAL).
Remark 1.7.7 (Check your work for linear functions). If \( f(x) = mx + b \) then \( \delta = \frac{\varepsilon}{|m|} \) is a good choice.

Remark 1.7.8 (\( \delta \) for quadratic functions). First check if you have a _____________, otherwise start with ___________.

Example 1.7.9. Given \( \varepsilon = 0.1 \), find the largest value of \( \delta \) in the proof of \( \lim_{x \to 5} (x^2 - 10x + 41) = 16 \).

Example 1.7.10. Prove that \( \lim_{x \to 2} x^2 = 4 \) using the formal definition of a limit.
Example 1.7.11. Setup the absolute value inequalities in the proof that \( \lim_{x \to \pi/2} \sin x = 1 \).
1.8 Continuity

**Section Objective(s):**
- Detect when a function is continuous and when it is discontinuous.
- Use continuity to quickly evaluate limits.
- Explain and classify the different types of discontinuities.
- Understand the statement of the Intermediate Value Theorem
- Apply the Intermediate Value Theorem to mathematically prove two functions intersect on a set interval.

**Definition(s) 1.8.1.** A function \( f \) is \( \lim_{x \to a} f(x) = f(a) \) if and only if

**Note additional subtleties:** This requires

1. \( f(a) \) to be defined
2. \( \lim_{x \to a} f(x) \) to exist.

**Definition(s) 1.8.2.** A function \( f \) is continuous on an interval if it is continuous at each point in that interval.

**Theorem 1.8.3.** Any \( f \) is continuous whenever it is \( \) in that interval.

**Theorem 1.8.4.** Trigonometric functions and root functions are also continuous everywhere in their domain.
Example 1.8.5. Evaluate \( \lim_{x \to 0} \frac{x^2 + 3x - 4}{x^2 - 1} \)

Example 1.8.6. Consider the graph of \( f(x) \) to the right on the domain \([-2, 5]\). Determine where the function is continuous.

Theorem 1.8.7. If \( f \) and \( g \) are continuous at \( a \) and \( c \) is constant, then the following are also continuous at \( a \).

1. \( f + g \)
2. \( f - g \)
3. \( cf \)
4. \( fg \), if \( g(a) \neq 0 \)

Theorem 1.8.8. If \( g \) is continuous at ___ and \( f \) is continuous at ______, then the composite function \( f \circ g \) given by \( (f \circ g)(x) = f(g(x)) \) is _______________________.

18
Definition(s) 1.8.9. If \( f \) is defined near \( a \) (except perhaps at \( a \)), we say that \( f \) is
\( \) (or \( f \) has a \( \) at \( a \)) if \( f \) is not
continuous at \( a \).

Definition(s) 1.8.10.

1. A \( \) is a discontinuity that can be
\( \) by redefining \( f \) at just a \( \).

2. An \( \) another name for a vertical asymptote.

3. A \( \) is where the function is discontinuous because it
\( \) from one value \( \).

Pictures

Example 1.8.11. \( f(x) = \frac{x^3 - 1}{x - 1} \) has a removable discontinuity at \( x = 1 \). Find a function \( g \) that agrees with
\( f \) for \( x \neq 1 \) and is continuous at 1.
Theorem 1.8.12 (Intermediate Value Theorem (IVT)). Suppose that $f$ is continuous on the closed interval $[a, b]$ and let $N$ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.

Remark 1.8.13. The intermediate value theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$.

Example 1.8.14. Use the Intermediate Value Theorem to show that there is a root of the function:

$f(x) = x^4 + x - 3$ on the interval $(1, 2)$. 


Example 1.8.15. Explain why \( f(x) = \begin{cases} \frac{-2}{3-x} & \text{if } x \neq 3 \\ 3 & \text{if } x = 3 \end{cases} \) is discontinuous at \( x = 3 \). Write the largest interval on which \( f(x) \) is continuous.

Example 1.8.16. Show that \(|x|\) is continuous everywhere.
Example 1.8.17. Explain why the function $f(x) = \sqrt{1 + \frac{1}{x}}$ is continuous at every number in its domain. State the domain of $f$.

Example 1.8.18. Prove that the equation $\cos x = x^3$ has at least one solution. What interval is it in?
Chapter 2

Derivatives
Forest for the Trees

Now that we have a good understanding about the subtitles of limits we can now formally compute slope at a point or even figure out the slope at every point on the graph. In sections 1 and 2 we will do just that.

Starting in section 3 we learn tricks for calculating the derivative (slope at a point) for complicated functions. For these sections (3, 4, and 5) we can pretend we are Batman adding new tools and weapons to our utility belt.

In section 6, 7, and 8 we unleash our new found tools to help us defeat some real world problems and help us get ready for chapter 3, Applications of Differentiation.
2.1 Derivatives and Rates of Change

Section Objective(s):

- Use limits to calculate slopes of tangent lines or instantaneous velocity
- Calculate the derivative of a function at a specific point
- Determine tangent line equations

Definition(s) 2.1.1. The \textbf{derivative} of a function \( f \) at a number \( c \), denoted by \( f'(c) \), is the number

\[
f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}
\]

if the limit exists. An equivalent formulation is

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]

Remark 2.1.2. The tangent line to the graph of \( y = f(x) \) at the point \( (c, f(c)) \) is the line through \underline{ } whose slope is equal to \underline{ }.

Remark 2.1.3. If \( f(t) \) measures distance of a moving object, and \( t \) is time, then the \underline{ } (or \underline{ } of the moving object, denoted \( v(t) \), is the limit of the \underline{ } (as defined in Section 1.4).

\[
v(t) =
\]
Example 2.1.4. Explain why $f(x) = |x|$ is not differentiable at $x = 0$.

Example 2.1.5. Are the following functions differentiable at $x = 2$? Why or why not?

(a) $f(x) = \begin{cases} x & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$
(b) \( f(x) = \begin{cases} 
    x^2 & \text{if } x < 2 \\
    4x - 4 & \text{if } x \geq 2 
\end{cases} \)

(c) \( f(x) = \begin{cases} 
    x^2 & \text{if } x < 2 \\
    4 & \text{if } x \geq 2 
\end{cases} \)
Example 2.1.6. Assuming a function $f(x)$ is differentiable at $x = c$, come up with a general equation for the tangent line to $f$ at $c$. 
2.2 The Derivative as a Function

Section Objective(s):
• Given a function sketch the graph of its derivative.
• Calculate the formula for the derivative given a function.
• Investigate how a function can fail to be differentiable.
• Associate higher derivatives with acceleration and jerk.

Definition(s) 2.2.1. The _________________________________, denoted by ____, is the function

Another common notation is to write $\frac{df}{dx}$ or $\frac{d}{dx}f(x)$ instead of $f'(x)$. The derivative at $x = c$ in this notation is written $\frac{df}{dx} \bigg|_{x=c}$.

Definition(s) 2.2.2. A function $f$ is ________________________ if $f'(c)$ exists. It is ____________________________ if it is differentiable at every number in $(a, b)$.

Theorem 2.2.3. If $f$ is differentiable at $c$, then $f$ is continuous at $c$.

How Can a Function Fail to be Differentiable (at a point $c$)?

• $f$ is __________________________ at $c$
CHAPTER 2. DERIVATIVES

- \( \lim_{h \to 0^-} \frac{f(x + h) - f(x)}{h} \neq \lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h} \)

- \( \lim_{x \to c} |f'(x)| = \infty \)

---

**Definition(s) 2.2.4.** The \( \underline{\text{second derivative}} \) of \( f \) is the derivative of \( f'(x) \), denoted by \( f''(x) \) or \( \frac{d^2f}{dx^2} \).

In general, the \( n^{\text{th}} \) derivative, denoted by \( f^{(n)}(x) \) or \( \frac{d^n f}{dx^n} \), is the derivative of \( f^{(n-1)}(x) \).

**Remark 2.2.5.** If \( f(t) \) measures position/distance of a moving object, then the \( \underline{\text{acceleration}} \) of the object, \( a(t) \), is the \( \underline{\text{second derivative}} \) of \( f \), and the first derivative of the velocity, \( v(t) \).

That is:
Example 2.2.6. Compute the derivative using the limit definition for the following functions:

(a) \( f(x) = 3x^2 + x - 8 \)

(b) \( f(x) = \frac{5}{x - 3} \)

(c) \( f(x) = \frac{1}{x^2} \)
Example 2.2.7. Is the function $f(x) = \begin{cases} 2x + 1 & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \geq 0 \end{cases}$ differentiable at $x = 0$? Why or why not?

Example 2.2.8. The graph of a function $f(x)$ is shown on the left. Use it to sketch a graph of $f'(x)$. 
2.3 Differentiation Formulas

Section Objective(s):
- Learn when to use each of the differentiation formulas
- Understand how select differentiation formulas can be proven.

**Theorem 2.3.1** (Constant, Linear, and Power Rules). Take \( c \) and \( n \) to be any constants

\[
\frac{d}{dx}(c) = 0
\]

\[
\frac{d}{dx}(x) = 1
\]

\[
\frac{d}{dx}(x^n) = nx^{n-1}
\]

**Theorem 2.3.2** (Constant Multiple, Sum, and Difference Rules). Take \( c \) to be a constant and \( f, g \) both differentiable functions, then:

\[
\frac{d}{dx}(cf(x)) = cf'(x)
\]

\[
\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)
\]

\[
\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)
\]
Theorem 2.3.3 (Product, Quotient). Take $f, g$ both differentiable functions, then:

$$
\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)
$$

Remark 2.3.4. The most powerful theorem we currently have is $\frac{d}{dx}(x^n) = nx^{n-1}$ (the power rule... pun intended). You will see that it gets used in nearly every problem. The proof of it is very complicated and is typically broken down into steps

1. $n$ is a positive integer (proof is in this section, need binomial theorem)
2. $n$ is a negative integer (proof in this section, need quotient rule)
3. $n$ is a rational number (need implicit differentiation, section 2.6)
4. $n$ is any real number (need knowledge of logarithmic differentiation, MTH133)

Example 2.3.5. Differentiate $\frac{x^2 + 2x - 3}{\sqrt{x}}$
Example 2.3.6. Find the equation of the tangent line to the curve $y = \frac{2x}{x + 1}$ at the point $(1, 1)$.

Example 2.3.7. For what values of $x$ does $f(x) = x^3 + 3x^2 + x + 3$ have a horizontal tangent line?
Example 2.3.8. Find the velocity and acceleration of the position function \( s(t) = \frac{3}{2-x} \).

Example 2.3.9. Find the equations of the tangent lines to the curve \( y = \frac{x-1}{x+1} \) that are parallel to the line \( x-2y = 2 \).

Example 2.3.10. Let \( f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ mx+b & \text{if } x > 2 \end{cases} \). Find the values of \( m \) and \( b \) that make \( f \) differentiable everywhere.
2.4 Derivatives of Trigonometric Functions

Section Objective(s):
- Prove some important trig limits
- Use these trig limits to prove trig derivatives

Theorem 2.4.1.

\[
\lim_{x \to 0} \frac{\sin x}{x} = \quad \lim_{x \to 0} \frac{\cos x - 1}{x} =
\]

Idea of Proof that \( \lim_{x \to 0} \frac{\sin x}{x} = \)
More generally,

**Theorem 2.4.2.** If $\lim_{x \to a} f(x) = 0$ then

$$\lim_{x \to a} \frac{\sin(f(x))}{f(x)} =$$

From these we get many new types of limit problems. Let’s try a few before moving on to derivatives.

**Example 2.4.3.** Find the limit $\lim_{x \to 0} \frac{\sin 5x}{x}$

**Example 2.4.4.** Find the limit $\lim_{x \to 0} \frac{\sin 2x}{x^2 + x}$

**Example 2.4.5.** Find the limit $\lim_{t \to 0} \frac{\tan 3t}{\sin t}$
Example 2.4.6. Find the limit \( \lim_{x \to 0} \frac{\cos x - 1}{\sin x} \)

Example 2.4.7. Find the limit \( \lim_{x \to 0} \frac{\sin(x^2)}{x} \)

Example 2.4.8. Find the limit \( \lim_{x \to 1} \frac{\sin(x - 1)}{x^2 + x - 2} \)

Theorem 2.4.9.

(a) \( \frac{d}{dx} (\sin(x)) = \)

(b) \( \frac{d}{dx} (\cos(x)) = \)
Before trying a formal proof. Let’s try to sketch the derivative of $\sin(x)$ and $\cos(x)$ to verify that this make intuitive sense.

Proof of (a).
Example 2.4.10. Derive formulas for the derivatives of the following functions.

(a) **Put to memory:** \( \frac{d}{dx} (\tan(x)) \)

(b) **Put to memory:** \( \frac{d}{dx} (\sec(x)) \)

(c) **Extra:** \( \frac{d}{dx} (\csc(x)) \)

(d) **Extra:** \( \frac{d}{dx} (\cot(x)) \)
CHAPTER 2. DERIVATIVES

2.5 The Chain Rule

Section Objective(s):

- Identify when a function can be decomposed into a composition of two functions.
- Understand the statement of the chain rule and how to apply it.

Theorem 2.5.1. If \( g \) is differentiable at \( x \) and \( f \) is differentiable at \( g(x) \), then the composition \( F = f \circ g \) defined by \( F(x) = f(g(x)) \) is differentiable at \( x \) and \( F' \) is given by

\[
F'(x) = f'(g(x)) \cdot g'(x)
\]

In Leibniz notation, the same thing can be written as

\[
\frac{dF}{dx} = \frac{dF}{dg} \cdot \frac{dg}{dx}
\]

Remark 2.5.2. Although the Leibniz notation should not literally be thought of as a fraction, it still gives a good way to remember the chain rule: it looks like the \( dg \)'s just cancel.

Remark 2.5.3. If \( y = f(x) \) is differentiable, then by combining the chain rule and the power rule, we can differentiate powers of \( f \):

\[
\frac{d}{dx} (f(x)^n) = nf(x)^{n-1} f'(x)
\]

Or, written in terms of \( f(x) \), this would be

\[
\frac{d}{dx} (f(x)^n) = n f(x)^{n-1} f'(x)
\]
Example 2.5.4. Calculate the derivatives of the following functions by first writing them as a composition of simpler functions, and then applying the chain rule:

(a) \( f(x) = (7x^3 + 2x^2 - x + 3)^5 \)

(b) \( g(x) = \sin (3x^2 + 5x + 8) \)

(c) \( h(x) = x \tan (2\sqrt{x}) + 7 \)
Example 2.5.5. Prove the quotient rule by writing \( \frac{f(x)}{g(x)} \) as \( f(x) \cdot \frac{1}{g(x)} \) and using the product rule and chain rule.

Example 2.5.6. Recall that \( f \) is an even function if \( f(-x) = f(x) \) for all \( x \), and that \( f \) is an odd function if \( f(-x) = -f(x) \) for all \( x \). Use the chain rule to prove that the derivative of an even function is an odd function.
Example 2.5.7. A table of values for $f, g, f', \text{ and } g'$ is given:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$g(x)$</th>
<th>$f'(x)$</th>
<th>$g'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>8</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>2</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>

(a) If $h(x) = f(g(x))$, find $h'(1)$.

(b) If $H(x) = g(f(x))$, find $H'(1)$.

Example 2.5.8. The graphs of two functions, $f$ and $g$, are pictured below.

Let $h(x) = f(g(x))$, and $p(x) = g(f(x))$. Compute the following derivatives, or explain why they do not exist:

(a) $h'(2)$

(b) $h'(7)$

(c) $p'(1)$
Example 2.5.9. Calculate the derivatives of the following functions:

(a) \[ y = \sin(x \cos(x)) \]

(b) \[ y = \sqrt{x + \sqrt{x}} \]

(c) \[ F(x) = (4x - x^2)^{100} \]
Section Objective(s):
- Use differentiation to solve real world problems related to physics.

**Definition(s) 2.7.1.** The instantaneous rate of change of $y = f(x)$ with respect to $x$ is the slope of the tangent line (a.k.a. derivative). Using Leibniz notation, we write:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

As was brought up in Section 2.1, the units for $\frac{dy}{dx}$ are the units for $y$ divided by the units for $x$.

**Remark 2.7.2** (From Section 2.2). If $s = f(t)$ is the position function of an object that is moving in a straight line, then $v(t) = s'(t)$ represents the __________ at time $t$. Also, $a(t) = v'(t) = s''(t)$ is the __________ of the object at time $t$.

**Remark 2.7.3.** Some common phrases and their mathematical interpretations

- When is the object at rest (or stand still)?
- When is the object moving forward/backward?
- When is the speeding up/down?
- Find the total distance traveled
- (Gravity Problems) When does an object achieve its maximum height?
Example 2.7.4. A particle moves according to the law of motion \( s(t) = \cos(\pi t/4) \) with \( 0 \leq t \leq 10 \), where \( t \) is measured in seconds and \( s \) in feet.

(a) Find the velocity at time \( t \).

(b) What is the velocity after 3 seconds?

(c) When is the particle at rest?

(d) When is the particle moving in the positive direction?

(e) Find the total distance traveled in the first 8 seconds?

(f) When is the particle speeding up? When is it slowing down?
Example 2.7.5. The height (in meters) of a projectile shot vertically upward from a point 15 m above ground level with an initial velocity of 10 m/s is $h = 15 + 10t - 5t^2$ after $t$ seconds.

(a) Find the velocity after 2 seconds.

(b) When does the projectile reach its maximum height?

(c) What is the maximum height?

(d) When does it hit the ground?

(e) With what velocity does it hit the ground?
Example 2.7.6 (FS14 E1). The figure below shows the velocity \( v(t) \) of a particle moving on a horizontal coordinate line, for \( t \) in a closed interval \([0, 10]\).

Fill in the following blanks. No partial credit available. No work needed. Use interval notation where appropriate.

(a) The particle is moving forward during: \[ \text{interval notation} \]

(b) The particle’s speed is increasing during: \[ \text{interval notation} \]

(c) The particle has positive acceleration during: \[ \text{interval notation} \]

(d) The particle has zero acceleration during: \[ \text{interval notation} \]

(e) The particle achieves its greatest speed at: \[ \text{interval notation} \]

(f) The particle stands still for more than an instant during: \[ \text{interval notation} \]
2.6 Implicit Differentiation

**Section Objective(s):**
- Recognize the need for implicit differentiation.
- Find the slopes of various curves by applying implicit differentiation.

**Definition(s) 2.6.1. Implicit Differentiation** is a method of differentiating both sides of an equation with respect to $x$ and then solving the resulting equation for $y'$.

**Remark 2.6.2.** Be careful that you are applying power, product, quotient, and chain rules correctly.

**Remark 2.6.3.** Khan Academy has 7 videos and practice problems all about implicit differentiation. Feel free to check them out at:


**Example 2.6.4.** Find $dy/dx$ of $x^3 + y^3 = 1$ using implicit differentiation.
Example 2.6.5. Use implicit differentiation to find an equation of the tangent line to the curve
\[ y \sin 2x = x \cos 2y \] at the point \((\pi/2, \pi/4)\).

Example 2.6.6. Find \(dy/dx\) by implicit differentiation of \(\tan(x - y) = \frac{y}{1 + x^2}\).
Example 2.6.7.

(a) Find $y'$ if $x^3 + y^3 = 6xy$.

(b) Find and equation of the tangent line at the point $(3,3)$.

(c) At what point in the first quadrant is the tangent line horizontal?
Remark 2.6.8. Let’s practice implicit differentiation with 3 variables: $x$, $y$, and $t$. Where $x = x(t)$ and $y = y(t)$, that is $x$ and $y$ depend on $t$. This will lead us nicely into related rates problems.

Example 2.6.9. Suppose you have the following information:

$$169 = x(t)^2 + y(t)^2 \quad y'(t) = -3 \quad x(a) = 5 \quad x(t) \geq 0 \quad y(t) \geq 0$$

Find $x'(a)$.
2.8 Related Rates

Section Objective(s):

- Translate sentences into mathematical equations.
- Apply implicit differentiation and the chain rule to solve many types of related rates problems.

Remark 2.8.1 (General Technique for solving Related Rate Problems).

(i) Relating given quantities with an ______________.

(ii) ______________________ the equation (using implicit differentiation and the chain rule).

(iii) Solving for some ______________ (derivative).

Remark 2.8.2. Geometric/Trig formulas you are expected to know

- **Circles**
  \[ C = \pi d \quad d = 2r \quad A = \pi r^2 \]

- **Triangles**
  \[ A = \frac{1}{2}bh \quad c^2 = a^2 + b^2 \quad \text{(for right } \Delta s) \]
  \[ \sin \theta = \frac{O}{H} \quad \cos \theta = \frac{A}{H} \quad \tan \theta = \frac{O}{A} = \frac{\sin \theta}{\cos \theta} \quad \text{(for right } \Delta s) \]

- **Rectangles (Squares)**
  \[ P = 2l + 2w \quad A = lw \]

- **Spheres**
  \[ V = \frac{4}{3}\pi r^3 \quad SA = 4\pi r^2 \]

- **Cones**
  \[ V = \frac{1}{3}(\text{area of base}) \]

- **Cylinders**
  \[ V = h(\text{area of base}) \]
Example 2.8.3. A 13 foot ladder is leaning against a wall, and is sliding down, with the top of the ladder moving downwards along the wall at 3 ft/sec. How fast is the bottom of the ladder moving away from the wall when the bottom is 5 ft from the base of the wall?

Example 2.8.4. A cone-shaped tank is filling with water at a constant rate of 9 ft$^3$/min. The tank is 10 ft tall, and has a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?
Example 2.8.5. A sphere is growing, its volume increasing at a constant rate of 10 in$^3$ per second. Let $r(t)$, $V(t)$, and $S(t)$, be the radius, volume, and surface area of the sphere at time $t$. If $r(1) = 2$, then compute:

(a) $r'(1)$

(b) $S'(1)$

Example 2.8.6. A circle is growing, its radius (in inches) given by $r(t) = \sqrt{t}$ for $t$ in seconds. How fast is the area growing at time $t = 4$?
Example 2.8.7. A balloon is rising vertically above a field, and a person 500 feet away from the spot on the ground underneath the balloon is watching it, measuring the angle of inclination, $\theta$. When the angle is $\pi/4$ radians, the angle is increasing at $\frac{1}{10}$ radians per minute. At that moment, how fast is the balloon rising?

Example 2.8.8. Let $f(x) = x - x^2 = x(1 - x)$, and let $\theta$ be the angle between the positive $x$-axis and the line joining the point $(x, f(x))$ with the origin. At what rate is $\theta$ changing, with respect to $x$, when $x = 1 - \frac{1}{\sqrt{3}}$?
Example 2.8.9. A person is lifting a weight with a pulley. The pulley is 25 feet off the ground, the rope is 30 feet long, and the person is holding the end of the rope 5 feet off the ground. If the person is walking backwards (away from the pulley) at a constant rate of 5 ft/sec, how fast is the weight rising when the person is 10 feet from the spot on the ground directly under the pulley?

Example 2.8.10. A spotlight on the ground shines on a wall 12 meters away. If a man 2 meters tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?
2.9 Linear Approximations and Differentials

**Section Objective(s):**
- Utilize the tangent line or differentials to estimate how a function is changing around a specific point.

**Remark 2.9.1.** Linear approximation and tangent line approximation are two names for using the equation of a tangent line to approximate a function

**Definition(s) 2.9.2.**

\[
L(x) = f(x) + f'(a)(x - a)
\]

is called the linearization of \( f \) at \( a \).

**Note:** compare this to the equation of a tangent line through \((a, f(a))\):

\[
y - f(a) = f'(a)(x - a)
\]

**Remark 2.9.3.** An equivalent notion to linearization is differentials. Consider the definitions:

\[
dy = f'(x)dx
\]

Where \( dy \) and \( dx \) are considered variables in their own right.

**Definition(s) 2.9.4.**

\[
\Delta x = x - a \quad \Delta y = f(x + \Delta x) - f(x)
\]

**Remark 2.9.5.**

\[
\approx \quad \approx
\]
Example 2.9.6.

(a) Find the linearization $L(x)$ of the function $f(x) = \sqrt{x}$ at $a = 9$

(b) Use the linearization to approximate $\sqrt{10}$

Example 2.9.7.

(a) Find the differential $dy$ of $y = \cos(\pi x)$

(b) Evaluate $dy$ for $x = 1/3$ and $dx = -0.02$
Example 2.9.8.

(a) Find the linearization $L(x)$ of the function $f(x) = \sin x$ at $a = \pi/4$

(b) Use the linearization to approximate $\sin(11\pi/40)$

Example 2.9.9.

(a) Find the differential $dy$ of $y = \sqrt{x^2 + 8}$

(b) Evaluate $dy$ for $x = 1$ and $dx = 0.02$
Example 2.9.10. Use linear approximation to estimate $\sqrt{1001}$

Example 2.9.11. Use linear approximation to estimate $\frac{1}{4.002}$
Chapter 3

Applications of Differentiation
Forest for the Trees

In Chapter 3 we apply our new found differentiation skills on various applications. The two main topics are curve sketching (3.5) and optimization problems (3.7). Once we have mastered these sections we will be able to solve problems such as:

Example: Give a detailed sketch of: \( y = \frac{x}{\sqrt{x^2 + 1}} \)

Example: A rain gutter is to be constructed from a metal sheet of width 30cm by bending up one-third of the sheet on each side through an angle \( \theta \). How should \( \theta \) be chosen so that the gutter will carry the maximum amount of water?

Some more minor topics for this chapter will include: The Mean Values Theorem, Newton’s Method, and Anti-Derivatives.
3.1 Maximum and Minimum Values

**Section Objective(s):**
- Explain the Extreme Value Theorem.
- Use the Closed Interval Method to identify absolute maxima and minima.

**Definition(s) 3.1.1.** Let \( c \) be a number in the domain \( D \) of a function \( f \). Then \( f(c) \) is the

- **absolute (global) maximum** value of \( f \) on \( D \) if \( f(c) \geq f(x) \) for all \( x \) in \( D \).

- **absolute (global) minimum** value of \( f \) on \( D \) if \( f(c) \leq f(x) \) for all \( x \) in \( D \).

- **local maximum** value of \( f \) on \( D \) if \( f(c) \geq f(x) \) for all \( x \) near \( c \).

- **local maximum** value of \( f \) on \( D \) if \( f(c) \leq f(x) \) for all \( x \) near \( c \).

Maximums and minimums are often referred to as **extreme values**.

**Pictures:**

**Remark 3.1.2.** The book uses “near \( c \)” to mean technically that the statement is true in some **open** interval containing \( c \). Sometimes these definitions can make you crazy.
CHAPTER 3. APPLICATIONS OF DIFFERENTIATION

**Theorem 3.1.3 (Extreme Value Theorem (EVT)).** If \( f \) is continuous on a \( [a, b] \), then \( f \) attains an absolute maximum value \( f(c) \) and an absolute minimum value \( f(d) \) at some numbers \( c \) and \( d \) in \( [a, b] \).

**Definition(s) 3.1.4.** A \( \underline{\text{critical number}} \) of a function \( f \) is a number \( c \) in the domain of \( f \) such that either \( f'(c) = 0 \) or \( f'(c) \) does not exist.

**Remark 3.1.5.** If \( f \) has a local maximum or minimum at \( c \), then \( c \) is a \( \underline{\text{critical number}} \) of \( f \).

**Theorem 3.1.6.** To find the absolute maximum and minimum values of a continuous function \( f \) on a closed interval \([a, b]\):

1. Find the values of \( f \) at the \( \underline{\text{critical numbers}} \) of \( f \) in \((a, b)\).
2. Find the values of \( f \) at \( \underline{\text{endpoints}} \) \( (a \text{ and } b) \).
3. The largest of the values from above is the absolute \( \underline{\text{maximum}} \) value; the smallest is the absolute \( \underline{\text{minimum}} \) value.

**Remark 3.1.7.** In **Section 3.3** we will find a way to classify local maximums and minimums on any domain! (not just closed intervals)

**Example 3.1.8.** Find the absolute maximum and minimum values of \( f(x) = x^3 - 12x + 1 \) on the interval \([1, 4]\)
Example 3.1.9. Sketch a graph of a function \( f \) that is continuous on \([1, 5]\) and has all of the following properties

- An absolute minimum at 2
- An absolute maximum at 3
- A local minimum at 4

Example 3.1.10. Sketch a graph of a function \( f \) who has domain \([-2, 4]\) that has an absolute maximum but no local maximum.
Example 3.1.11. Find the critical numbers for the function \( f(x) = \frac{x - 1}{x^2 + 4} \).

Example 3.1.12. Find the absolute maximum and minimum values of \( f(t) = 2 \cos t + \sin(2t) \) on the interval \([0, \pi/2]\).
Section Objective(s):

- State the Mean Value Theorem and draw pictures to help us understand its meaning.
- Identify points on the correct interval that satisfy the Mean Value Theorem.

**Theorem 3.2.1 (Rolle’s Theorem).** Let \( f(x) \) be a function which satisfies the following three properties:

1. \( f(x) \) is ______________________ the interval \([a, b]\)
2. \( f(x) \) is ______________________ \((a, b)\)
3. \( f(a) = f(b) \)

Then there is a number \( c \) in \((a, b)\) such that ____________________.

**Remark 3.2.2.** The conclusion of Rolle’s Theorem says that if the function values agree at the endpoints, then there is a place in between where the tangent line is horizontal.
Theorem 3.2.3 (Mean Value Theorem (MVT)). Let \( f(x) \) be a function which satisfies the following two properties:

1. \( f(x) \) is continuous on the interval \([a, b]\)
2. \( f(x) \) is differentiable on \((a, b)\)

Then there is a number \( c \) in \((a, b)\) such that

\[
 f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Remark 3.2.4. The conclusion of The Mean Value Theorem says that there is a place in the interval where the tangent line is \underline{ \hspace{2cm} } to the \underline{ \hspace{2cm} } line between the endpoints.

Remark 3.2.5. Notice that Rolle’s Theorem and The Mean Value Theorem tell you that “there exists” a number \( c \) with certain properties, but neither theorem tells you what that the value of \( c \) is, or how to find it.

Picture:

Corollary 3.2.6. If \underline{ \hspace{2cm} } for all \( x \) in an interval \((a, b)\), then \( f(x) \) must be constant on \((a, b)\).

Corollary 3.2.7. If \underline{ \hspace{2cm} } for all \( x \) in an interval \((a, b)\), then \( f(x) = g(x) + c \) for some constant \( c \).
Example 3.2.8. For each of the following functions (on the given domain), tell whether the hypotheses of the Mean Value Theorem are satisfied. If so, try to find the value of $c$ guaranteed by the theorem.

(a) $f(x) = |x|$ on the domain $[-3, 3]$

(b) $f(x) = x^2 + 3x + 5$ on the domain $[0, 1]$

(c) $f(x) = \frac{1}{x}$ on the domain $[1, 3]$
Example 3.2.9. Prove that the equation $x^3 + x + 1 = 0$ has exactly one solution.
3.3 Derivatives and Graphs

Section Objective(s):

- Utilize the derivative to determine when a function is increasing or decreasing.
- Use the second derivative to determine when a function is concave up or down.

Theorem 3.3.1.

(a) If $f'(x) > 0$ on $(a, b)$, then $f(x)$ is ________________ on $(a, b)$.

(b) If $f'(x) < 0$ on $(a, b)$, then $f(x)$ is ________________ on $(a, b)$.

Theorem 3.3.2 (First Derivative Test). Suppose that $f(x)$ is a function and that $c$ is a ________________ (or “_______________”) of $f(x)$.

(a) If $f'(x)$ changes from ________________ at $x = c$, then $f(x)$ has a local ________________ at $x = c$.

(b) If $f'(x)$ changes from ________________ at $x = c$, then $f(x)$ has a local ________________ at $x = c$.

(c) If $f'(x)$ does not ________________ at $x = c$, then $f(x)$ has ________________ a local maximum nor a local minimum at $x = c$.

Pictures:
Example 3.3.3. For the following functions, find the intervals on which it is increasing and decreasing, and find where the local maximum and local minimum values occur.

(a) \( f(x) = 2x^3 + 3x^2 - 36x \) on the domain \((-\infty, \infty)\)

(b) \( f(x) = \frac{x}{x^2 + 1} \) on the domain \((-\infty, \infty)\)

(c) \( f(x) = \cos^2(x) - 2\sin(x) \) on the domain \([0, 2\pi]\)
Definition(s) 3.3.4. If the graph of $f$ lies above all of its tangents on an interval $I$, then it is called \underline{concave up} on $I$. If the graph of $f$ lies below all of its tangents on $I$, it is called \underline{concave down} on $I$.

Picture:

Theorem 3.3.5 (Concavity Test).

(a) If \underline{$f''(x) > 0$} for all $x$ in $I$, then the graph of $f$ is concave upward on $I$.

(b) If \underline{$f''(x) < 0$} for all $x$ in $I$, then the graph of $f$ is concave downward on $I$.

Definition(s) 3.3.6. A point $P$ on a curve $y = f(x)$ is called an \underline{inflection point} if $f$ is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at $P$.

Theorem 3.3.7 (Second Derivative Test). Suppose $f''$ is continuous near $c$.

(a) If $f'(c) = 0$ and \underline{$f''(c) > 0$}, then $f$ has a local minimum at $c$.

(b) If $f'(c) = 0$ and \underline{$f''(c) < 0$}, then $f$ has a local maximum at $c$. 
Example 3.3.8. Suppose that the graph to the right is of is of $f$.
For which values of $x$ does $f$ have an inflection point?

Example 3.3.9. Suppose that the graph to the right is of is of $f'$.
For which values of $x$ does $f$ have an inflection point?

Example 3.3.10. Suppose that the graph to the right is of is of $f''$.
For which values of $x$ does $f$ have an inflection point?

Example 3.3.11. Find where $f(x) = x\sqrt{6 - x}$ is concave up and where it is concave down. Where are the inflection points?
Example 3.3.12. Find where \( f(x) = x - 4\sqrt{x} \) is concave up and where it is concave down. Where are the inflection points?

Example 3.3.13. Use the second derivative test to classify the critical points for \( f(x) = \frac{x^2}{x - 1} \).
3.4 Limits at Infinity; Horizontal Asymptotes

Section Objective(s):
- Recognize horizontal asymptotes of a function given graphically.
- Investigate horizontal asymptotes of a function given algebraically by using limits at infinity.

Definition(s) 3.4.1. Let \( f \) be a function defined on some interval \((a, \infty)\). Then \( \lim_{x \to \infty} f(x) = L \) means that the values of \( f(x) \) can be made arbitrarily close to \( L \) by taking \( x \)
________________________.

Definition(s) 3.4.2. Let \( f \) be a function defined on some interval \((-\infty, a)\). Then \( \lim_{x \to -\infty} f(x) = L \) means that the values of \( f(x) \) can be made arbitrarily close to \( L \) by taking \( x \)
________________________.

Definition(s) 3.4.3. The line \( y = L \) is called a __________________________ of the curve \( y = f(x) \) if either

\[ \lim_{x \to -\infty} \frac{1}{x^r} = 0 \]

or

\[ \lim_{x \to \infty} \frac{1}{x^r} = 0 \]

Theorem 3.4.4. If \( r > 0 \) is a rational number, then

If \( r > 0 \) is a rational number such that \( x^r \) is defined for all \( x \), then

\[ \lim_{x \to \infty} \frac{1}{x^r} = 0 \]

or

\[ \lim_{x \to -\infty} \frac{1}{x^r} = 0 \]
Example 3.4.5. Find the limits or show that they do not exist

(a) \( \lim_{x \to \infty} \frac{2x - 1}{5x + 3} \)

(b) \( \lim_{x \to \infty} x \sin \frac{1}{x} \)

(c) \( \lim_{x \to \infty} \frac{\sqrt{x} - x^2}{2x + x^2} \)

(d) \( \lim_{x \to -\infty} \frac{x^2}{\sqrt{3x^4 + 1}} \)
3.5 Curve Sketching

**Section Objective(s):**
- Summarize all of our current algebra and calculus knowledge to sketch accurate graphs of functions.

**Remark 3.5.1** (General Guidelines for Curve Sketching).

When trying to sketch the graph of a given function \( f(x) \), ask yourself the following:

1. What is the __________ of \( f \)?

2. What are the \( x \) and \( y \) __________ of the graph?

3. Does the graph have any __________ (is \( f \) an ______ function? an ______ function? neither?)
   a) ______ if _________________ for all \( x \)
   b) ______ if _________________ for all \( x \)

4. Does the graph have any asymptotes (horizontal, vertical, slant)?
   a) Has horizontal asymptote \( y = c \) if \( \lim_{x \to \infty} f(x) = c \) or if \( \lim_{x \to -\infty} f(x) = c \).
   b) Has vertical asymptote \( x = c \) if either of \( \lim_{x \to c^+} f(x) \) or \( \lim_{x \to c^-} f(x) \) are equal to either \( \infty \) or \( -\infty \).
   c) Has _________________ \( y = mx + b \) if
      
      or if

5. Where is \( f(x) \) increasing and decreasing?
   a) See **Theorem 3.3.1**

6. Where are the local maximum and minimum points?
   a) Use **First Derivative Test** or **Second Derivative Test**
7. Where is the graph concave up and where is it concave down?

   a) See Theorem 3.3.5

8. Where are the inflection points?

   a) Find where \( f''(x) \) changes sign

**Example 3.5.2.** Sketch graphs of the following:

(a) \( y = x(x - 4)^3 \)
(b) \( y = \frac{x^3}{x^2 + 1} \)
(c) \( y = \frac{x}{\sqrt{x^2 + 1}} \)
(d) \( y = \sin^2(x) \) on the interval \([0, 2\pi]\)
3.7 Optimization Problems

Section Objective(s):

- Analyze real world problems and transform statements into mathematical equations.
- Apply our maxima/minima knowledge to help solve optimization problems.

Remark 3.7.1. Related rates problems are to __________________________ as optimization problems are to ___________ and ___________ problems.

Example 3.7.2. Find the maximum area of a rectangle inscribed in an equilateral triangle of side length 6 and one side of the rectangle lies along the base of the triangle.
Theorem 3.7.3 (Steps in Solving Optimization Problems).

1. Understand the problem.
   - Read the problem through in its entirety
   - Determine what is given and what is unknown

2. Draw a __________
   - This is useful in most problems

3. Introduce __________
   - Assign symbols to what needs maximized or minimized.
   - Select symbols for other quantities and label the diagram when appropriate.

4. Find an __________ that relates the quantities with what needs to be maximized/minimized

5. Use _______________ to reduce down to one variable (when applicable)

6. Use the methods in 3.1/3.3 to find an absolute maximum/minimum. In particular, if the domain is closed then the Closed Interval Method in Section 3.1 can be used.

Example 3.7.4. Find a point on the line $y = 2x + 3$ that is closest to the origin.
Example 3.7.5. A fish tank with a square base is to be made of glass sides, plastic on the base, and an open top. The fish tank needs to hold 5 cubic feet of water. Glass costs $3 per square foot and plastic cost $2 per square foot. What is the cheapest the tank can cost?

Example 3.7.6. A right circular cylinder is inscribed in a right circular cone of radius 2 and height 3. Find the largest possible volume of such a cylinder.
Example 3.7.7. A right circular cylinder is inscribed in a sphere of radius 2. Find the largest possible volume of such a cylinder.

Example 3.7.8. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?
Example 3.7.9. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle $\theta$. How should $\theta$ be chosen so that the gutter will carry the maximum amount of water?
3.8 Newton’s Method

Section Objective(s):
- Recognize how and why Newton’s Method finds intersections between functions.
- Analyze how Newton’s Method may fail.

Remark 3.8.1. Newton’s method is an extremely powerful technique used in finding roots of equations - in general the convergence is quadratic: as the method converges on the root, the difference between the root and the approximation is squared (the number of accurate digits roughly doubles) at each step. However, there are some difficulties with the method which we will discuss below. Newton’s method can be altered to help approximate solutions to many equations.

Pictures of the idea of Newton’s Method

Theorem 3.8.2 (Newton’s Method). If $x_1$ is the initial guess of some root of $f(x)$ then
Remark 3.8.3. Newton’s method may fail to converge to an answer or may find the wrong root. See the pictures below for how this can happen. When this occurs it can usually be fixed by selecting an alternative initial guess ($x_1$).

<table>
<thead>
<tr>
<th>Horizontal tangent</th>
<th>Never converges</th>
<th>Converges to wrong root</th>
</tr>
</thead>
</table>

Example 3.8.4. Approximate the root of $f(x) = x^3 + x + 4$ by taking the initial guess $x_1 = -1$.

Compute $x_2$ and $x_3$. *Don’t simplify $x_3$.*
Example 3.8.5. Use Newton’s method to approximate the positive root of \( x^2 = \sin x \). Take \( x_1 = \pi/3 \). Calculate \( x_2 \).

Example 3.8.6. Use Newton’s method to approximate \( \sqrt[3]{29} \). Calculate up to \( x_3 \). Do not simplify.
Example 3.8.7. Approximate $\sqrt{5}$ using

(a) Linearization with $a = 4$

(b) Newton’s method with $x_1 = 2$. Compute $x_2$ and $x_3$. 
3.9 Antiderivatives

Section Objective(s):
- Compute general antiderivatives for many types of functions.
- Solve initial value problems for particular antiderivative functions.
- Use antiderivatives to calculate velocity or position from acceleration.

Definition(s) 3.9.1. A function $F$ is called an antiderivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

Theorem 3.9.2. If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is

$$F(x) + C$$

where $C$ is an arbitrary constant.

Example 3.9.3. Find the most general antiderivative of the functions below.

(a) $f(x) = \pi + \frac{1}{x^2}$

(b) $f(x) = \sqrt{x}(6 + 7x)$
Definition(s) 3.9.4. A ____________________________ is an equation involving the derivatives of an unknown function.

Definition(s) 3.9.5. An ____________________________ is a differential equation for \( y = f(x) \) along with an ____________________________, such as \( f(c) = a \) for some constants \( c \) and \( a \). The solution to the initial value problem is a solution to the differential equation that also satisfies the initial condition.

Remark 3.9.6. In an initial value problem, if the unknown function \( f(t) \) represents position as a function of time, then the initial condition \( f(t_0) = a \) means that at time \( t = t_0 \), the object is at position \( a \). If the unknown function represents velocity, then the same initial condition means the velocity is \( a \) at time \( t = t_0 \).

Example 3.9.7. Below is a graph of \( f' \). Sketch the graph of \( f \) if \( f \) is continuous and \( f(0) = -2 \)

![Graph of f']
Example 3.9.8. A particle is moving with velocity $v(t) = \sin t - \cos t$, and has initial position $s(0) = 0$. Find the position function of the particle.

Example 3.9.9. Solve the initial value problem

$$f''(x) = 1 + 3\sqrt{x}, \quad f(4) = 25$$
Example 3.9.10. Solve the initial value problem

\[ f''(x) = 2 + \cos(x), \quad f(0) = -1, \quad f(\pi/2) = 0 \]
Chapter 4

Integrals
Forest for the Trees

In Chapters 4 and 5 we will bring up the second main topic: area ‘under’ a function. We will see that surprisingly this is related to derivatives. We will learn how to find the area ‘under’ more complicated functions and the area trapped between two functions. Additionally we will learn how to apply this to real world problems in physics. Once we have mastered these sections we will be able to solve problems such as:

Example: An object moves along a line with velocity $v(t) = t(9 - t)$ inches per second. At time $t = 0$, the object is 2 inches to the right of the origin. How far did the object move during those 15 seconds?

Example: Find the area of the region bounded by the graphs of $y = 5x - x^2$ and $y = x$.

Example: Find the area between the curve $x \cos(x^2)$ and the $x$-axis between the vertical lines $x = 0$ and $x = \sqrt{\pi}$.
4.1 Areas and Distances

Section Objective(s):
- Estimate the area under a curve using rectangles with heights given by left endpoints or right endpoints.
- Solve for over/under estimates for the area under a curve using rectangles.

The goal of this section is to understand how to approximate areas with rectangles. Suppose that we want to approximate the area between the graph of a continuous function $f(x)$ and the $x$-axis between $x = a$ and $x = b$ (suppose for now that $f$ is positive). Let’s sub-divide the interval $[a, b]$ into $n$ sub-intervals $[x_0, x_1]$ through $[x_{n-1}, x_n]$ of equal width $\Delta x$. If we pick a point $x_i^*$ in each interval $[x_i, x_{i+1}]$, then we can estimate the area under the graph by the sum of the areas of the rectangles with width $\Delta x$ and height $f(x_i^*)$:

$$\text{Area} \approx f(x_0^*)\Delta x + f(x_1^*)\Delta x + \cdots + f(x_{n-1}^*)\Delta x$$

Definition(s) 4.1.1.
- An ________________ is when the $x_i^*$ are all chosen to be the global maximum on $[x_i, x_{i+1}]$ for each $i$.
- A ________________ is when the $x_i^*$ are all chosen to be the global minimum on $[x_i, x_{i+1}]$ for each $i$.
- A ________________ is when $x_i^*$ is the left endpoint of $[x_i, x_{i+1}]$ for each $i$.
- A ________________ is when $x_i^*$ is the right endpoint of $[x_i, x_{i+1}]$ for each $i$.

Remark 4.1.2. To get better approximations of area, use ____________________.
Remark 4.1.3. If the velocity of an object is constant, then we have

\[ \text{distance} = \ \text{velocity} \times \text{time} \]

We can think of this product as being the area of a rectangle of width “time” and height “velocity”. Thought of in this way, the distance traveled by an object from time \( t = a \) to \( t = b \) is given by the area under the graph of the velocity from \( t = a \) to \( t = b \).

Example 4.1.4. Approximate the area under the graph of \( f(x) = \sqrt{x} \) from \( x = 0 \) to \( x = 4 \) using 4 rectangles of equal width, using a:

(a) left-hand sum

(b) right-hand sum
Example 4.1.5. The speed of a runner (in ft/s) is given in the table below at different times (in seconds).

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(t)$</td>
<td>0</td>
<td>6.2</td>
<td>10.8</td>
<td>14.9</td>
<td>18.1</td>
<td>19.4</td>
<td>20.2</td>
</tr>
</tbody>
</table>

(a) Give an upper estimate of the distance traveled by the runner.

(b) Give a lower estimate of the distance traveled by the runner.

Example 4.1.6. Below is the graph of the velocity of a car (in ft/s) as it is coming to a stop.

Estimate the distance the car travels as it comes to a stop using the various types of sums.
Appendix E - Sigma Notation

Section Objective(s):
- Express sums using sigma notation.
- Memorize a few common finite sums.
- Understand basic properties of finite sums and use them to compute more complicated finite sums.

Definition(s) 4.1.7. If $a_m, a_{m+1}, \ldots, a_{n-1}, a_n$ are real numbers and $m$ and $n$ are integers such that $m \leq n$, then
\[
\sum_{i=m}^{n} a_i =
\]

Definition(s) 4.1.8. This way of short-handing sums of many numbers is called _____________ (uses the Greek letter $\Sigma$ “Sigma”). The letter $i$ above is called the ___________________________ and it takes on consecutive integer values starting with $m$ and ending with $n$.

Theorem 4.1.9. If $c$ is any constant then:

(a) $\sum_{i=m}^{n} ca_i =$

(b) $\sum_{i=m}^{n} (a_i + b_i) =$

(c) $\sum_{i=m}^{n} (a_i - b_i) =$

Theorem 4.1.10. Let $c$ be a constant and $n$ a positive integer. Then

(a) $\sum_{i=1}^{n} 1 =$

(b) $\sum_{i=1}^{n} i =$

(c) $\sum_{i=1}^{n} i^2 =$
Example 4.1.11. Evaluate the following sums

(a) \[ \sum_{k=0}^{4} \frac{2k - 1}{2k + 1} \]

(b) \[ \sum_{i=0}^{4} (2 - 3i) \]

(c) \[ \sum_{i=1}^{38} (3^i - 3^{i-1}) \]
Example 4.1.12. Write the sum: $\sqrt{3} + \sqrt{4} + \cdots + \sqrt{25}$ in sigma notation

Example 4.1.13. Write the sum: $\sqrt{3} - \sqrt{5} + \sqrt{7} - \sqrt{9} + \cdots + \sqrt{27}$ in sigma notation
4.2 The Definite Integral

**Section Objective(s):**
- Use the limit of finite sums to calculate the definite integral of a function.
- Identify how the definite integral relates with area under the curve.
- Understand intuitively the properties of definite integrals.

**Theorem 4.2.1** (Definite Integral). If \( f \) is integrable on \([a, b]\), then

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

where

\[
\Delta x = \frac{b - a}{n}
\]

and

\[
x_i = a + i \Delta x
\]

**Remark 4.2.2.** The definite integral, \( \int_{a}^{b} f(x) \, dx \) gives the ______________ between the curve \( f \) and the \( x \)– axis.
Example 4.2.3. Use the definition of the definite integral to verify that \( \int_{1}^{3} \left( \frac{x + 1}{2} \right) \, dx = 3 \)

Example 4.2.4. Evaluate the definite integral \( \int_{1}^{7} f(x) \, dx \).

Where \( f(x) \) is given by the function to the right.
Example 4.2.5. Evaluate the definite integral \( \int_{-1}^{4} (x^2 - x + 1) \, dx \)
**Theorem 4.2.6** (Properties of Definite Integrals).

1. \( \int_a^a f(x) \, dx = 0 \)

2. \( \int_a^b f(x) \, dx = \int_b^a f(x) \, dx \)

3. \( \int_a^b [cf(x) + dg(x)] \, dx = c \int_a^b f(x) \, dx + d \int_a^b g(x) \, dx \)

4. \( \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx \)

5. If \( f(x) \geq g(x) \) for \( a \leq x \leq b \), then \( \int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx \).

6. \( \int_a^b c \, dx = c(b - a) \)

**Example 4.2.7.** Suppose

\[ \int_1^3 f(x) \, dx = 3 \quad \int_1^5 f(x) \, dx = 10 \]

Find \( \int_3^5 [2f(x) + 1] \, dx \)
4.3 The Fundamental Theorem of Calculus

Section Objective(s):

- Relate slopes and areas through the two parts of the Fundamental Theorem of Calculus.
- Develop the idea of a proof for the two parts of the Fundamental Theorem of Calculus.
- Use the chain rule and properties of definite integrals so solve more complicated problems.
- Use the antiderivative to calculate definite integrals.

Theorem 4.3.1 (FTC, Part 1). If $f$ is continuous on $[a, b]$, then the function $g$ defined by

$$g(x) = \int_{a}^{x} f(t) \, dt, \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on $(a, b)$, and $g'(x) =$ ________

Remark 4.3.2. Here is an idea of the proof:
Theorem 4.3.3 (FTC, Part 2). If \( f \) is continuous on \([a, b]\), then
\[
\int_{a}^{b} f(x) \, dx =
\]
where \( F \) is any antiderivative of \( f \), that is, a function such that \( F' = f \).

Remark 4.3.4. Here is an idea of the proof:

Remark 4.3.5. The two parts of the FTC together state that differentiation and integration are ____________ processes.

Remark 4.3.6. From our perspective FTC, Part ___ is the most important because it allows us to calculate definite integrals without having to take limits of Riemann sums!
Example 4.3.7. Evaluate the following integrals

(a) \[ \int_{1}^{9} \sqrt{x} \, dx \]

(b) \[ \int_{-1}^{5} (u + 1)(u - 2) \, du \]

(c) \[ \int_{1}^{2} \frac{x^4 + 1}{x^2} \, dx \]

(d) \[ \int_{-1}^{2} |x| \, dx \]
Example 4.3.8. Find the derivatives of the following functions:

(a) \( f(x) = \int_{1}^{x} t^2 \, dt \)

(b) \( g(x) = \int_{x}^{2} \sqrt{t^2 + 3} \, dt \)

(c) \( H(x) = \int_{\tan x}^{\tan^2 x} \frac{1}{\sqrt{2 + t^4}} \, dt \)

Example 4.3.9. Identify what is wrong with the evaluation:

\[
\int_{-1}^{1} \frac{1}{x^2} \, dx = \left[ -x^{-1} \right]_{-1}^{1} \\
= \left[ -1^{-1} + (-1)^{-1} \right] = [-1 - 1] = -2
\]
4.4 Indefinite Integrals and Net Change

Section Objective(s):

- Use the Net Change Theorem to continue calculating definite integrals.
- Given a velocity function and a set time interval find the total distance traveled.

Definition(s) 4.4.1. An \(_{\text{ }}\) of \(f\), written \(\int f(x) \, dx\), is an \(_{\text{ }}\) of \(f\). In other words,

\[
F(x) = \int f(x) \, dx \quad \text{means}
\]

Theorem 4.4.2. (Linearity Properties of Indefinite Integrals)

For functions \(f(x)\) and \(g(x)\), and any constant \(k \in \mathbb{R}\),

\[
\int [f(x) + g(x)] \, dx =
\]

and

\[
\int k f(x) \, dx =
\]

Theorem 4.4.3. (Net Change Theorem)

The net change of a quantity \(F(x)\) from \(x = a\) to \(x = b\) is the integral of its derivative from \(a\) to \(b\).

That is, if \(F'(x) = f(x)\), then

\[
= \int_a^b F'(x) \, dx = \int_a^b f(x) \, dx
\]

In other words,

\[
= F(a) + \int_a^b F'(x) \, dx = F(a) + \int_a^b f(x) \, dx
\]
Remark 4.4.4. If \( s(t) \) represents the position of a moving object, and \( v(t) \) the velocity, then the **Net Change Theorem** says we can compute the net displacement of the object from time \( t = a \) to time \( t = b \) by integrating the velocity:

\[
\int_a^b s'(t) \, dt = \int_a^b v(t) \, dt
\]

**Example 4.4.5.** Find the most general function \( F(x) \) with the property that:

(a) \( F'(x) = \frac{7}{x^7} - \frac{5}{x^4} + 18 \)

(b) \( F'(x) = \frac{1}{\sqrt{x}} + \cos(x) \)

(c) \( F'(x) = \sec(x) (\tan(x) + \sec(x)) \)
Example 4.4.6. An object moves along a line with velocity $v(t) = t(9 - t)$ inches per second. At time $t = 0$, the object is 2 inches to the right of the origin.

(a) What is the position of the object at time $t = 15$ seconds?

(b) How far did the object move during those 15 seconds?

Example 4.4.7. Find the net change of the function $f(x)$ from $x = 0$ to $x = 1$, if $f'(x) = x^2 + x + 1$. 

4.5 The Substitution Rule

Section Objective(s):
- Develop a substitution rule to find antiderivatives of more complicated functions.
- Use properties of definite integrals and symmetric functions to create a few more theorems.

Theorem 4.5.1. If \( u = g(x) \) is a differentiable function whose range is an interval \( I \) and \( f \) is continuous on \( I \), then

\[
\int f(g(x)) g'(x) \, dx = \int f(u) \, du
\]

Remark 4.5.2. The Substitution Rule says essentially that is is permissible to operate with \( dx \) and \( du \) after the integral signs as if they were differentials.

Theorem 4.5.3. If \( g'(x) \) is continuous on \([a, b]\) and \( f \) is continuous on the range of \( u = g(x) \), then

Remark 4.5.4. This change in the limits of integration is annoying and somewhat unnecessary right now (so long as you change your variable back) however in Calc II once you start doing trig substitutions it becomes extremely useful. WeBWorK will force you to practice changing the limits of integration so we will practice here too.

Definition(s) 4.5.5. Recall again that \( f \) is called ______ if \( f(-x) = \) ______

and called ______ if \( f(-x) = \) ______.

Theorem 4.5.6. Suppose \( f \) is continuous on \([-a, a]\) then:

(a) If \( f \) is even, then \( \int_{-a}^{a} f(x) \, dx = \)

(b) If \( f \) is odd, then \( \int_{-a}^{a} f(x) \, dx = \)
Example 4.5.7. Evaluate the following indefinite integrals:

(a) \[ \int x^2 \sqrt{x^3 + 1} \, dx \]

(b) \[ \int \frac{dt}{(1 - 3t)^5} \]

(c) \[ \int \sec^3 x \tan x \, dx \]

(d) \[ \int \frac{\sin t}{\cos^2(\cos(t))} \, dt \]
Example 4.5.8. Evaluate the following definite integrals:

(a) \[ \int_0^3 \frac{x}{\sqrt{1 + 5x}} \, dx \]

(b) \[ \int_{-\pi/4}^{\pi/4} (1 + x + x^2 \tan x) \, dx \]
Example 4.5.9. Evaluate the following definite integrals by changing the limits of integration appropriately:

(a) \[ \int_{0}^{1} \sqrt{1 + 7x} \, dx \]

(b) \[ \int_{0}^{\sqrt{\pi}} x \cos(x^2) \, dx \]
Chapter 5

Applications of Integration
5.1 Area Between Curves

Section Objective(s):

- Express the area bounded by two curves as a definite integral and evaluate.
- Identify when it is advantageous to integrate with respect to $y$ instead of $x$.

**Theorem 5.1.1.** If $f(x)$ and $g(x)$ are continuous functions on $[a, b]$ where $f(x) \geq g(x)$ for all $x$ in $[a, b]$, then the area of the region in between the graphs of $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is given by

$$\text{Area} = \int_{a}^{b} (f(x) - g(x)) \, dx$$

**Remark 5.1.2.** This theorem only applies if $f(x) \geq g(x)$ on $[a, b]$. In the more general case (where the graphs cross), we can use the following theorem.

**Theorem 5.1.3.** If $f(x)$ and $g(x)$ are continuous functions on $[a, b]$, then the area of the region in between the graphs of $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is given by

$$\text{Area} = \int_{a}^{b} |f(x) - g(x)| \, dx$$

**Remark 5.1.4.** If it is more convenient, you can think of $x$ as a function of $y$, and integrate with respect to $y$. For example, to find the area between the graphs of $x = f(y)$ and $x = g(y)$ between $y = a$ and $y = b$, compute

$$\text{Area} = \int_{a}^{b} |f(y) - g(y)| \, dy$$
Example 5.1.5. Find the area between the graphs of $y = -x$ and $y = \cos(x)$ between $x = 0$ and $x = \frac{\pi}{2}$.

Example 5.1.6. Find the area enclosed by the curves $y = \sec^2(x)$ and $y = 8\cos(x)$ on the interval $[-\frac{\pi}{3}, \frac{\pi}{3}]$.

Example 5.1.7. Find the area of the region bounded by the graphs of $y = 5x - x^2$ and $y = x$. 
Example 5.1.8. Find a positive value of $c$ so that the area between $y = x^2 + 1$ and $y = x - c$
from $x = 0$ to $x = 1$ is equal to 1.

Example 5.1.9. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. 


5.5 Average Value of a Function

Section Objective(s):

- Calculate the average value of a function over an interval.
- Create a Mean Value Theorem for Integrals.

**Definition(s) 5.5.1.** The average value of a function \( f(x) \) on the interval \([a, b]\) is defined to be:

\[
\text{ave} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

**Theorem 5.5.2.** (Mean Value Theorem for Integrals)

If \( f \) is continuous on \([a, b]\), then there exists a number \( c \) in \([a, b]\) such that

\[
f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

In other words, there exists a number \( c \) in \([a, b]\) such that

**Example 5.5.3.** Find the average value of the following functions on the given interval:

(a) \( f(x) = \sqrt{x} \) on \([0, 4]\)

(b) \( f(x) = \sin(x) \) on \([0, \pi]\)
Example 5.5.4. Find the average value of the given function on the given interval. Also, find the value of $c$ guaranteed by the Mean Value Theorem for Integrals.

(a) $f(x) = \sqrt[3]{x}$ on $[0, 8]$

(b) $f(x) = \sec^2(x)$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$