Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. Compute the *derivatives* of the following functions: (**DO NOT SIMPLIFY**)

(a) (7 points) $f(x) = x^2 \cos x - \sqrt[4]{x}$

Solution:

$$\begin{aligned} f'(x) &= (x^2 \cos x)' - (x^{\frac{1}{4}})' \\ &= 2x\cos x + x^2(-\sin x) - \frac{1}{4}x^{-\frac{3}{4}} \end{aligned}$$

(b) (7 points) $g(x) = \dfrac{x}{\tan(3x-1)}$

Solution:

Quotient Rule:

$$\begin{aligned} g'(x) &= \frac{(x)'\tan(3x-1) - x\,(\tan(3x-1))'}{\tan^2(3x-1))} \\ &= \frac{(1)\tan(3x-1) - x\,\sec^2(3x-1)\cdot(3)}{\tan^2(3x-1))} \end{aligned}$$

Product Rule: Write $g(x) = x\,(\tan(3x-1))^{-1}$. Then,

$$\begin{aligned} g'(x) &= (x)'(\tan(3x-1))^{-1} + x\,((\tan(3x-1))^{-1})' \\ &= (1)(\tan(3x-1))^{-1} + x\,(-1) + x\,(\tan(3x-1))^{-2}\sec^2(3x-1)(3) \end{aligned}$$

⋆

2. (7 points) A snow ball melts so that its surface area decreases at a rate of 2 cm^2/min. How fast is its radius decreasing when the radius is 5 cm? (*Include units.*)

Solution:

Set $S(t) =$ The surface area of the snow ball at time t.

Data: $\frac{dS}{dt}(t) = -2\ \text{cm}^2/\text{min}$

Question: $\frac{dr}{dt}(t_0) = ?$ where $r(t_0) = 5$ cm.

Relation between $S(t)$ and $r(t)$: $\boxed{S(t) = 4\,\pi\, r^2}$

$$\frac{dS}{dt}(t) = 8\,\pi\, r(t)\,\frac{dr}{dt}(t)$$

At $t = t_0$

$$\frac{dS}{dt}(t_0) = 8\,\pi\, r(t_0)\,\frac{dr}{dt}(t_0)$$

$\Longleftrightarrow$

$$-2 = 8\,\pi\,(5)\,\frac{dr}{dt}(t_0)$$

$\Longleftrightarrow$

$$\frac{dr}{dt}(t_0) = \frac{-1}{20\pi}\ \text{cm/min}$$

The radius is decreasing by $\frac{1}{20\pi}$ cm/min

3. (7 points) Using the Intermediate Value Theorem prove that the following equation has a solution

$$\sqrt[3]{x} + x^2 = \cos \pi x.$$

Solution:

Set $f(x) = \sqrt[3]{x} + x^2 - \cos \pi x$.

f is continuous everywhere

$f(0) = -1 < 0 < f(1) = 3$

By The IVT, there must be at least one $c \in (0, 1)$ such that $f(c) = 0$.

4. Calculate the following limits or show that they do not exist:

(a) (4 points) $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x^2+3}-2} =$

Solution:

$$\begin{aligned}\lim_{x\to 1} \frac{x-1}{\sqrt{x^2+3}-2} &= \lim_{x\to 1} \frac{x-1}{\sqrt{x^2+3}-2} \cdot \left(\frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2}\right) \\ &= \lim_{x\to 1} \frac{x-1}{x^2-1} \cdot \left(\sqrt{x^2+3}+2\right) \\ &= \lim_{x\to 1} \frac{1}{(x+1)} \cdot \left(\sqrt{x^2+3}+2\right) = \frac{1}{2}\cdot 4 = 2\end{aligned}$$

(b) (4 points) $\lim_{x\to 2^-} \dfrac{x\left(x^2-4\right)}{|x^2-4|} =$

Solution: For $x \to 2^-$ we know that $|x^2-4| = -(x^2-4)$. Therefore

$$\begin{aligned}\lim_{x\to 2^-} \frac{x\left(x^2-4\right)}{|x^2-4|} &= \lim_{x\to 2^-} \frac{x\left(x^2-4\right)}{-(x^2-4)} \\ &= \lim_{x\to 2^-} \frac{x}{-1} = -2\end{aligned}$$

5. (6 points) Consider the function $f(x) = \dfrac{1}{2x+1}$. Use the <u>limit definition</u> of the derivative to show that $f'(x) = \dfrac{-2}{(2x+1)^2}$. (Your calculation must include computing a limit.)

Solution: First lets simplify the difference quotient

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{2x+2h+1} - \frac{1}{2x+1}}{h} &= \frac{(2x+1)-(2x+2h+1)}{(2x+2h+1)(2x+1)} \cdot \frac{1}{h} \\ &= \frac{-2h}{(2x+2h+1)(2x+1)} \cdot \frac{1}{h} = \frac{-2}{(2x+2h+1)(2x+1)}\end{aligned}$$

From here we can evaluate quickly

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{-2}{(2x+2h+1)(2x+1)} = \frac{-2}{(2x+1)(2x+1)} = \frac{-2}{(2x+1)^2}$$

6. Suppose you are on the moon and you shoot an arrow straight upward. Its height in meters after t seconds is $s(t) = -3t^2 + 2t + 1$.

   (a) (2 points) Find the velocity at time t.

   Solution:

   $$v(t) = -6t + 2$$

   (b) (4 points) With what velocity (include units) will the arrow hit the ground?

   Solution: Let's first find when the arrow will hit the ground by solving $s(t) = 0$.

   $$-3t^2 + 2t + 1 = 0$$
   $$(3t + 1)(1 - t) = 0$$

   So $t = 1$ is the only positive time when arrow hits ground. Therefore

   $$v(1) = -6 + 2 = -4\text{m/s}$$

7. (8 points) Find the equation of the tangent line to the curve defined by the equation $\boxed{x\ \sin(x+y) = y - \pi}$ at the point (π, π).

Solution: Taking the derivative (with respect to x) on both sides and then plugging in $x = \pi$ and $y = \pi$ we get:

$$\begin{aligned} x\ \sin(x+y) &= y - \pi \\ \sin(x+y) + x\cos(x+y)\cdot(1+y') &= y' \\ \sin(2\pi) + \pi\cos(2\pi)\cdot(1+y') &= y' \\ \pi(1+y') &= y' \\ \pi &= y' - \pi y' \\ \frac{\pi}{1-\pi} &= y' \end{aligned}$$

Therefore an equation of the tangent line can be given by

$$\boxed{y - \pi = \frac{\pi}{1-\pi}(x - \pi)}$$

Multiple Choice. Circle the single best answer. No work needed. No partial credit available.

8. (4 points) Given a continuous function f on $(-\infty, \infty)$ with $f(0) = 6$, $f(2) = 1$, and $f(9) = -2$. Which of the following statements is necessarily true?

   A. There is c in [0,2] with $f(c) = 0$.

   B. There is c in [-1,5] with $f(c) = 0$.

   C. There is c in $[0, 2]$ with $f(c) = 3$.

   D. There are a and b in $(-\infty, \infty)$ with $a \neq b$ and f(a)=f(b).

   E. None of the above is necessarily true.

9. (4 points) Suppose $f(x)$ is continuous and differentiable and that $f'(x) < 0$ for all x, and $f(1) = 4$. What is true about $f(0)$?

   A. It is possible that $f(0) = 4$.

   B. It must be that $f(0) < 4$.

   C. It must be that $f(0) > 4$.

   D. There is not enough information.

10. (4 points) The domain of $f(x) = \sqrt{\dfrac{x}{x+2}}$ is:

    A. $(-\infty, -2) \cup (-2, 0]$

    B. $(-\infty, -2) \cup [0, \infty)$

    C. $(-\infty, -2)$

    D. $(-2, 0]$

    E. $(-2, \infty)$

11. (4 points) Suppose

$$f(2) = -3,\ g(2) = 5,\ f'(2) = 2,\ g'(2) = 6.$$

Then the derivative of $\dfrac{2g(x)}{1+f(x)}$ at $x = 2$ is

A. -11

B. 6

C. 4

D. 44/9

E. 22

12. (4 points) Calculate the derivative of $f(x) = \tan(\sin(x^2))$,

A. $f'(x) = -\sec^2(\sin(x^2))\cos(x^2)\,(2x)$

B. $f'(x) = \sec^2(\cos(2x))$

C. $f'(x) = (\sec^2 x)\cos(x^2)\,(2x)$

D. $f'(x) = \sec^2(\sin(x^2))\cos(x^2)\,(2x)$

E. $f'(x) = -(\sec^2 x)\cos(x^2)\,(2x)$

13. (4 points) For what value of c the function $f(x) = \begin{cases} x^2 - 10, & x \le c \\ 10x - 35, & x > c \end{cases}$ is continuous everywhere.

A. $c = 10$

B. $c = 5$

C. $c = \sqrt{10}$

D. $c = 20$

E. None of the above

14. (4 points) Evaluate: $\lim_{x \to 0} \frac{\sin(x^2 + 7x)}{2\,x}$

A. 0

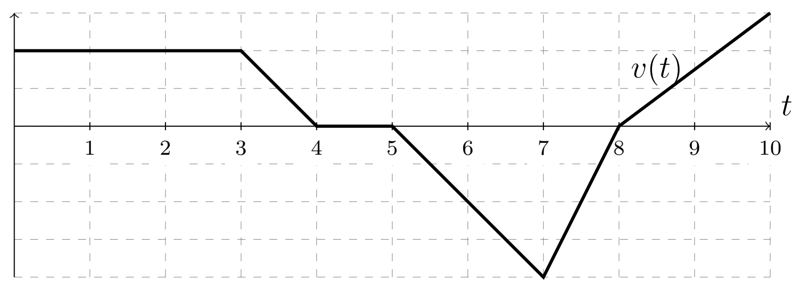
B. $\frac{7}{2}$

C. 1

D. 7

E. Does not exist

The following graph shows **the velocity** of a particle moving in a straight line for $t \in [0, 10]$.
Use it to answer the following two questions



15. (4 points) For what values of t is the particle moving forward?

A. (7, 10)

B. $(0, 3) \cup (4, 5)$

C. $(5, 8)$

D. $(3, 4) \cup (5, 7)$

E. $(0, 4) \cup (8, 10)$

16. (4 points) Which of the following statements is true of the particle on the time interval $(5, 7)$?

A. It is moving forwards and slowing down.

B. It is moving forwards and speeding up.

C. It is moving backwards and slowing down.

D. It is moving backwards and speeding up.

E. None of the above.

More Challenging Question(s).

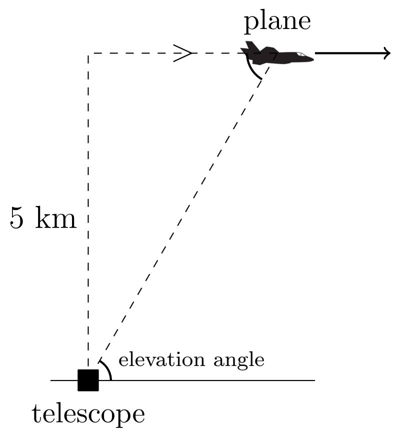
17. (4 points) Calculate the following limit: $\lim_{x\to 0^+}\left[x \sin\frac{1}{x}\right] =$

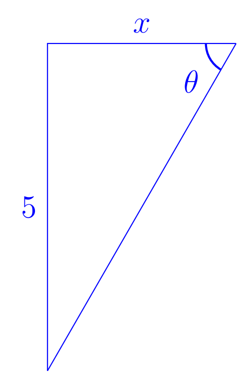
Solution: Since $-1 \le \sin\frac{1}{x} \le 1$, for $x > 0$, we have that $-x \le x\sin\frac{1}{x} \le x$. And because

$$\lim_{x\to 0^+} -x = \lim_{x\to 0^+} x = 0$$

we can use the Squeeze Theorem to show that $\lim_{x\to 0^+}\left[x \sin\frac{1}{x}\right] = \boxed{0}$

18. (10 points) A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\frac{\pi}{3}$, this angle is decreasing at a rate of $\frac{\pi}{6}$ rad/min. How fast is the plane traveling at that time?





Solution: Consider the simplified triangle above with notation. The planes speed can be given by $x'(a)$ where $t = a$ is this special time when $\theta(a) = \pi/3$.

$$\tan(\theta(t)) = \frac{5}{x(t)} \quad \text{(differentiate both sides)}$$

$$\sec^2(\theta(t))\cdot\theta'(t) = \frac{-5}{[x(t)]^2}\cdot x'(t) \quad \text{(plug in } t = a\text{)}$$

$$\sec^2\left(\tfrac{\pi}{3}\right)\cdot\frac{-\pi}{6} = \frac{-5}{[x(a)]^2}\cdot x'(a) \quad \text{(use } \tan\pi/3 = 5/x(a) \text{ to solve for } x(a)\text{)}$$

$$4\cdot\frac{-\pi}{6} = \frac{-5}{[5/\sqrt{3}]^2}\cdot x'(a) \quad \text{(algebra)}$$

$$\frac{10\pi}{9}\ \text{km/min} = x'(a)$$