Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. (4 points) Find the most general antiderivative of $f(x) = 2\sec^2(x) - x^5$.

Solution:

$$\int 2\sec^2(x) - x^5 \; dx = 2\tan(x) - \frac{1}{6}x^6 + C$$

2. (4 points) Evaluate: $\displaystyle\int_1^4 \frac{t^3 + \sqrt{t}}{t^2} \, dt$.

Solution:

$$\begin{aligned}
\int_1^4 \frac{t^3 + \sqrt{t}}{t^2} \, dt &= \int_1^4 t + t^{-3/2} \; dt \\
&= \left[\frac{1}{2}t^2 - 2t^{-1/2}\right]_1^4 \\
&= \left[\frac{1}{2}(16) - 2\left(\frac{1}{2}\right)\right] - \left[\frac{1}{2} - 2\right] \\
&= [7] - \left[-\frac{3}{2}\right] = \boxed{17/2}
\end{aligned}$$

3. (6 points) Let $F(x) = \displaystyle\int_{\cos x}^1 \sqrt{5 - t^2} \, dt$. Find $F'(x)$ ***without actually finding*** $\mathbf{F}(\mathbf{x})$.

Solution:

$$\begin{aligned}
F(x) &= \int_{\cos x}^1 \sqrt{5 - t^2} \; dt \\
F(x) &= -\int_1^{\cos x} \sqrt{5 - t^2} \; dt \\
F'(x) &= -\sqrt{5 - \cos^2 x} \cdot (-\sin x) \\
F'(x) &= \boxed{\sin x \cdot \sqrt{5 - \cos^2 x}}
\end{aligned}$$

4. (7 points) Find the absolute maximum and the absolute minimum values of

$$f(x) = \sin x + \cos x \qquad \text{on the interval } [0, \pi].$$

Solution:

$$f'(x) = \cos x - \sin x$$

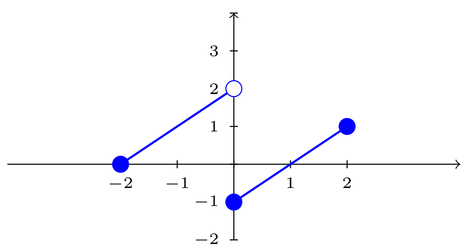
so the only critical point of f on the interval $[0, \pi]$ occurs at $x = \pi/4$ at which $f' = 0$. Testing all the critical points and the end points we see that

$$f(0) = 0 + 1 = 1$$
$$f(\pi/4) = \sqrt{2}/2 + \sqrt{2}/2 = \sqrt{2}$$
$$f(\pi) = 0 - 1 = -1$$

So the $\boxed{\text{absolute maximum is } \sqrt{2}}$ and the $\boxed{\text{absolute minimum is } -1}$.

5. Let $f(x) = \begin{cases} x + 2 & \text{if } -2 \le x < 0 \\ x - 1 & \text{if } 0 \le x \le 2 \end{cases}$

(a) (3 points) Sketch the graph of $f(x)$.



(b) (2 points) Use your sketch in part (a) to find the absolute maximum and absolute minimum values of f on $[-2, 2]$, if they exist.

Absolute Minimum=	-1
Absolute Maximum=	DNE

(c) (2 points) If an absolute extremum does not exist, explain why does this not contradict the statement of the Extreme Value Theorem.

Solution: The nonexistence of Absolute Maximum does not contradict the Extreme Value Theorem because the f is $\boxed{\text{not continuous}}$ on $[-2, 2]$.

6. Suppose:

$$f(x) = \frac{x^2}{x-1}, \qquad f'(x) = \frac{x(x-2)}{(x-1)^2}, \qquad f''(x) = \frac{2}{(x-1)^3}$$

(a) (1 point) What is the domain for f?

Solution: $(-\infty, 1) \cup (1, \infty)$

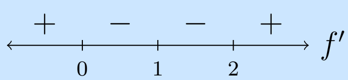
(b) (3 points) Write the equations for all vertical, horizontal, and slant asymptotes, if they exist.

Solution:

$$VA : x = 1$$
$$HA : NONE$$
$$SA : y = x + 1$$

(c) (4 points) Identify the intervals over which $f(x)$ is increasing / decreasing. *Write your answer in interval notation in the boxes below.*

Solution: Breaking up the real numbers at critical points and asymptotes and testing each subinterval we have



so therefore f is increasing on $(-\infty, 0) \cup (2, \infty)$ and decreasing on $(0, 1) \cup (1, 2)$

(d) (2 points) Identify all points (x, y) where f(x) attains its local maximum or minimum. *Write your answer in the boxes below.*

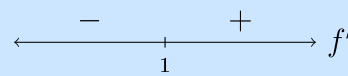
Solution:

f has a local minimum at $(x, y) = (2, 4)$

f has a local maximum at $(x, y) = (0, 0)$

(e) (2 points) Identify the intervals over which $f(x)$ is concave up / concave down. *Write your answer in interval notation in the boxes below.*

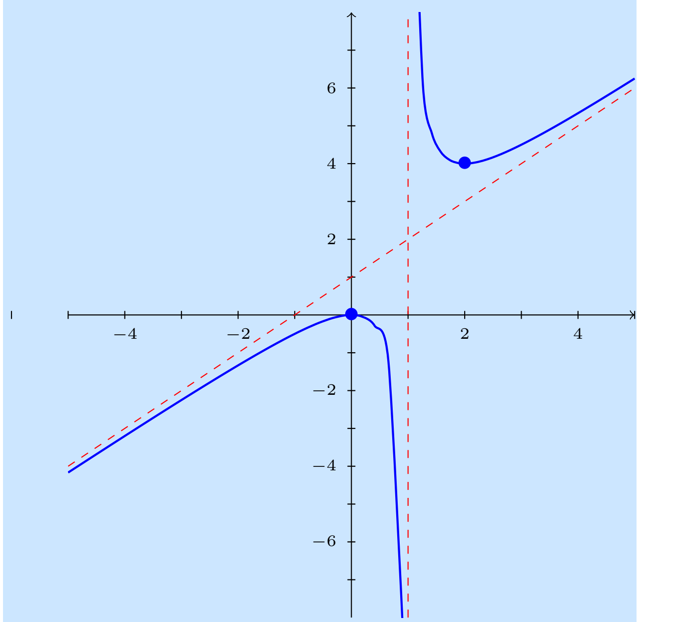
Solution: Breaking up the real numbers at possible inflection points and asymptotes and testing each subinterval we have



so therefore f is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$.

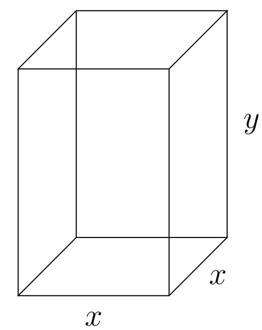
(f) (2 points) Sketch the curve of $y = f(x)$. *Parts (a)-(e) may be helpful.*

Solution:



7. A total of 225 ft^2 of material is used to make a box with **no top** and a **square** base.

(a) (6 points) Express the volume of the tank, V, only in terms of x, the length of the base.



Solution: From the fact that 225 ft^2 of material is to be used we can determine

$$\begin{aligned} x^2 + 4xy &= 225 \\ 4xy &= 225 - x^2 \\ y &= \frac{225 - x^2}{4x} \end{aligned}$$

Therefore

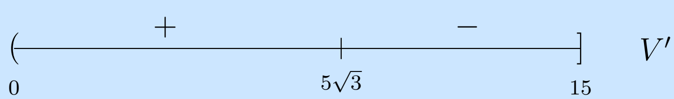
$$\begin{aligned} V &= x^2 \cdot y \\ V &= x^2 \cdot \left(\frac{225 - x^2}{4x}\right) \\ V &= \boxed{\frac{225x - x^3}{4}} \end{aligned}$$

(b) (8 points) Find the maximum volume of such a tank for $x \in (0, 15]$.
Include units! Use techniques of calculus to justify that your answer is a maximum.

Solution:

$$V' = \frac{225 - 3x^2}{4}$$

And so the only critical point of V in $(0, 15]$ is at $x = \sqrt{75} = 5\sqrt{3}$ (where $V' = 0$). We can then determine the sign of the derivative on this interval



and using the first derivative test we see that V is maximized when $x = 5\sqrt{3}$. So the maximum volume is

$$V = \boxed{\frac{225(5\sqrt{3}) - (5\sqrt{3})^3}{4} \text{ ft}^3}$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

8. (4 points) Newton's method can be used to approximate $\sqrt[5]{33}$ by finding the root of which of the following functions?

   A. $f(x) = x^5 + 33$

   B. $f(x) = \sqrt[5]{x} - 33$

   C. $f(x) = \sqrt[5]{x} + 33$

   D. $f(x) = x^5 - 33$

   E. $f(x) = x - (33)^5$

9. (4 points) Using three equally-spaced rectangles of equal width, find the upper sum approximation of the area between the curve $y = \dfrac{1}{x}$ and the $x-$axis from $x = 2$ to $x = 8$.

   A. $\frac{13}{12}$

   B. $\frac{11}{4}$

   C. $\frac{11}{6}$

   D. $\frac{39}{24}$

   E. 3

10. (4 points) Using a linear approximation with $a = 64$, find an estimate of $\sqrt{63}$

    A. $8 + \frac{1}{16}$

    B. $8 - \frac{1}{8}$

    C. $8 - \frac{1}{16}$

    D. $8 + \frac{1}{8}$

    E. 8

11. (4 points) Suppose $\int_{-1}^{2} f(x)\, dx = 6$ and $\int_{-1}^{3} f(x)\, dx = 8$. Find $\int_{3}^{2} 5f(x)\, dx$.

A. 70

B. -10

C. -14

D. 10

E. 2

12. (4 points) If the conclusion of the Mean Value Theorem is applied to the function $f(x) = \dfrac{1}{x-1}$ on the interval $[2, 5]$, which of the following values of c is correct?

A. $c = 1 + \frac{2}{\sqrt{3}}$

B. $c = \frac{7}{3}$

C. $c = 3$

D. c does not exist

E. None of the above.

13. (4 points) Which of these equations is the solution to the initial value problem

$$y' = \sin x, \quad y(0) = 2$$

A. $y = -\cos(x) + 3$

B. $y = -\cos(x) + 2$

C. $y = \cos(x) + 3$

D. $y = \cos(x) + 1$

E. $y = \cos(x)$

14. (4 points) Which of the following is the equation of a horizontal asymptote for the curve $y = \dfrac{3x + \sqrt{x}}{x(1-2x)}$?

A. $y = 0$

B. $y = 3$

C. $y = 3/2$

D. $y = -3/2$

E. $y = -3$

15. (4 points) $\displaystyle\int_0^3 |x - 1|\, dx =$

A. 3

B. $\frac{9}{2}$

C. $\frac{5}{2}$

D. 5

E. $\frac{7}{2}$

16. (4 points) Which of the following definite integrals is equivalent to the following limit of a Riemann sum?

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[3]{2 + \frac{4i}{n}} \cdot \frac{4}{n}$$

A. $\displaystyle\int_2^6 \sqrt[3]{2+x}\, dx$

B. $\displaystyle\int_2^6 \sqrt[3]{x}\, dx$

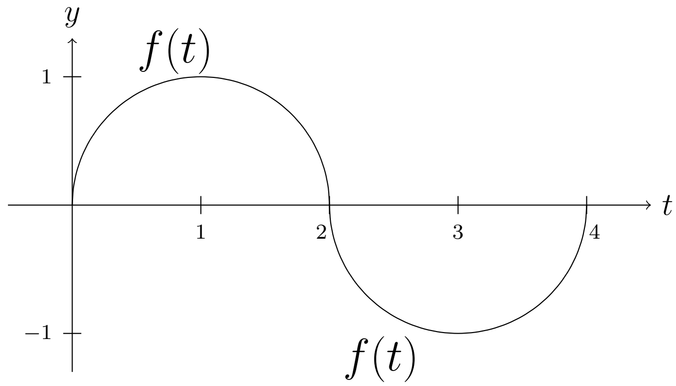
C. $\displaystyle\int_2^6 \sqrt[3]{2+4x}\, dx$

D. $\displaystyle\int_2^6 4\sqrt[3]{2+4x}\, dx$

E. None of the above.

More Challenging Question(s). Show all work to receive credit.

17. Below is the graph of the function $\mathbf{y} = \mathbf{f}(\mathbf{t})$ and is formed by two adjacent semi-circles of radius 1:



Consider the function $g(x)$ defined by $g(x) = \int_0^x f(t)\,dt$, for all $0 \leq x \leq 4$.

(a) (2 points) Calculate: $g(0) = 0$, $g(1) = \pi/4$, $g(2) = \pi/2$, $g(3) = \pi/4$ and $g(4) = 0$

(b) (2 points) List the critical point(s) of $g(x)$ and determine if each is a local minimum, a local maximum or neither for g.

Solution: $g'(x) = f(x)$ and so $g(x)$ has a critical point at $x = 2$. Since the derivative changes from positive to negative by the first derivative test we know this is a local maximum

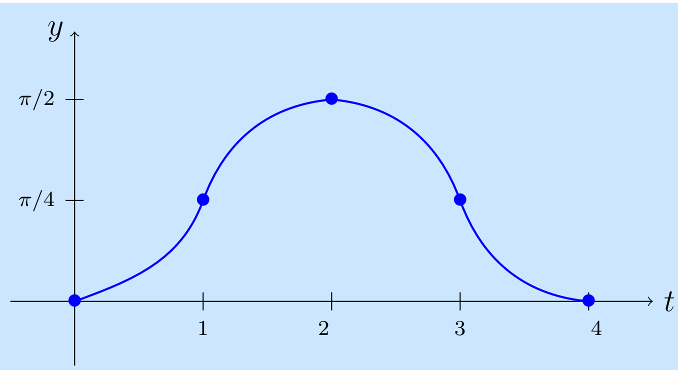
(c) (2 points) $g(2) = \frac{\pi}{2}$, and $g(0) = g(4) = 0.$

Thus, $g(x)$ achieves its global maximum in the interval $[0, 4]$ at $x = 2$

(d) (4 points) List the inflection points of $g(x)$.

Solution: Since $g''(x) = f'(x)$ then the inflection points of g are the local mins/maxes of f which occur at $(1, \frac{\pi}{4})$ and $(3, \frac{\pi}{4})$

(e) (4 points) Sketch the graph of $g(x)$.



Solution: