Name: _____ Signature: _____

Section: _____ Date: _____

- No calculators or notes allowed.
- The exam lasts exactly 90 minutes.
- No extra scratch paper: use backs of pages.
- If anything is unclear, raise your hand for an instructor.

Page	1	2	3	4	5	6	7	8	9	**Total**
Score										
Max	15	15	10	20	30	30	10	10	10	150

Fill in the Blank. Answers only: no work needed. No partial credit. (5 points each)

1. If $y = 2\ln(3\ln x)$, then the derivative $\dfrac{dy}{dx} =$ $\dfrac{2}{x\ln(x)}$

2. Evaluate the indefinite integral: $\displaystyle\int \frac{3x^2}{x^3+4}\,dx =$ $\ln(|x^3+4|)$ $+ C$

3. Evaluate the definite integral: $\displaystyle\int_{-1}^{2} \frac{1}{x^2}\,dx =$ ∞ or DNE

Fill in the Blank. Answers only: no work needed. No partial credit. (5 points each)

4. Let $f(x) = x^3 + x^2 + 8$, for $x > 0$, with $f(2) = 20$. Then the derivative $\left(f^{-1}\right)'(20) =$ 1/16

5. To hold a spring stretched 2 m beyond its natural length requires a force of 12 Newtons. The work needed to stretch the spring from 2 m beyond its natural length to 3 m beyond its natural length is: 15 joules.

6. Suppose constant money inflation makes prices increase exponentially. If prices double every 20 years, then prices will triple after $20\log_2(3)$ years.

Multiple choice. Circle the best answer. No partial credit. (5 points each)

7. Let $f(x) = (x{+}3)^x$ for $x > 0$. Then:
   (a) $f'(x) = x(x{+}3)^{x-1}$
   (b) $f'(x) = (x{+}3)^{x-1}\,(x + (x{+}3)\ln(x{+}3))$ $\longleftarrow$ Answer
   (c) $f'(x) = \ln(x{+}3)\ (x{+}3)^x$
   (d) $f'(x) = e^{x\ln(x+3)}$

8. Find which of the following functions is a solution of the differential equation $\dfrac{y'}{2} + \dfrac{y}{2} \;=\; e^{-3x}$.

   I. $y(x) \;=\; e^{-x} - e^{-3x}$ II. $y(x) \;=\; e^{-x} - 2e^{-3x}$ III. $y(x) \;=\; 2e^{-x} - e^{-3x}$

   (a) I only
   (b) II only
   (c) I and II
   (d) I and III $\longleftarrow$ Answer

Free response. For the remaining questions, **show all your work**, and $\boxed{\text{box}}$ your final answer.

9. (10+10=20 points) Evaluate the following limits

(a) $\lim_{x\to\infty} \frac{\ln^3 x}{\sqrt{x}}$

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln^3 x}{\sqrt{x}} &= \lim_{x\to\infty} \frac{3(1/x)\ln^2 x}{\frac{1}{2}x^{-1/2}} \\
&= \lim_{x\to\infty} \frac{6\ln^2 x}{\sqrt{x}} \\
&= \lim_{x\to\infty} \frac{12(1/x)\ln x}{\frac{1}{2}x^{-1/2}} \\
&= \lim_{x\to\infty} \frac{24\ln x}{\sqrt{x}} \\
&= \lim_{x\to\infty} \frac{24(1/x)}{\frac{1}{2}x^{-1/2}} \\
&= \lim_{x\to\infty} \frac{48}{\sqrt{x}} = \boxed{0}
\end{aligned}$$

(b) $\lim_{x\to 0^+} (2x)^{3x}$

Solution. Consider $\ln (2x)^{3x} = 3x\ln(2x)$

$$\begin{aligned}
\lim_{x\to 0^+} 3x\ln(2x) &= \lim_{x\to 0^+} \frac{3\ln(2x)}{1/x} \\
&= \lim_{x\to 0^+} \frac{3(1/x)}{-1/x^2} \\
&= \lim_{x\to 0^+} \frac{3x}{-1} = 0
\end{aligned}$$

Therefore $\lim_{x\to 0^+} (2x)^{3x} = e^{\ln(2x)^{3x}} = e^0 = \boxed{1}$

10. (10+10+10=30 points) Evaluate the following integrals

(a) $\displaystyle\int \frac{4}{\sqrt{9-t^2}}\ dt\ =$

Solution.

$$\begin{aligned}\int \frac{4}{\sqrt{9-t^2}}\ dt &= \frac{4}{3}\int \frac{1}{\sqrt{1-\left(\frac{t}{3}\right)^2}}\ dt \\ &= \frac{4}{3}\int \frac{1}{\sqrt{1-u^2}}\ (3du) \\ &= 4\sin^{-1} u = \boxed{4\sin^{-1}\left(\tfrac{t}{3}\right)+C}\end{aligned}$$

(b) $\displaystyle\int \frac{\cosh(\ln(4x))}{x}\ dx\ =$

Solution.

$$\begin{aligned}\int \frac{\cosh(\ln(4x))}{x}\ dx &= \int \frac{\cosh(u)}{x}\ dx \\ &= \int \cosh(u)\ du \\ &= \sinh(u)+C = \boxed{\sinh(\ln(4x))+C}\end{aligned}$$

(c) $\displaystyle\int x\ln(2x)\ dx\ =$

Solution. Consider integration by parts:

$$\begin{aligned} u &= \ln(2x) & dv &= x\ dx \\ du &= 1/x\ dx & v &= x^2/2\end{aligned}$$

Giving us:

$$\begin{aligned}\int x\ln(2x)\ dx &= \int u\ dv \\ &= uv - \int v\ du \\ &= x^2\ln(2x)/2 - \int x/2\ dx \\ &= \boxed{x^2\ln(2x)/2 - x^2/4 + C}\end{aligned}$$

11. (10+10+10=30 points) Evaluate the following integrals

(a) $\displaystyle\int \sin^3(x)\,\cos^4(x)\,dx\ =$

Solution.

$$\begin{aligned}\int \sin^3(x)\,\cos^4(x)\,dx &= \sin x \sin^2 x \cos^4 x\\ &= \sin x(1-\cos^2 x)\cos^4 x\\ &= \sin x\cos^4 x - \sin x \cos^6 x\\ &= \boxed{\frac{-1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C}\end{aligned}$$

(b) $\displaystyle\int \frac{1}{t^2\sqrt{1-t^2}}\,dt\ =$

Solution. Consider $t = \cos\theta \implies dt = -\sin\theta\ d\theta$ where $\theta \in [0,\pi]$.

$$\begin{aligned}\int \frac{1}{t^2\sqrt{1-t^2}}\,dt &= \int \frac{1}{\cos^2\theta\sqrt{1-\cos^2\theta}}(-\sin\theta\ d\theta)\\ &= \int \frac{1}{\cos^2\theta|\sin\theta|}(-\sin\theta\ d\theta) && \text{(Since } \theta\in[0,\pi]\text{)}\\ &= \int \frac{-1}{\cos^2\theta}\,d\theta\\ &= -\tan\theta\end{aligned}$$

Now since $t = \cos\theta \implies \sin\theta = \sqrt{1-t^2} \implies \tan\theta = \dfrac{\sqrt{1-t^2}}{t}$

$$= \boxed{-\frac{\sqrt{1-t^2}}{t} + C}$$

(c) $\displaystyle\int \frac{4x}{(x-1)(x^2+1)}\,dx\ =$

Solution. Consider

$$\begin{aligned}\frac{A}{x-1} + \frac{Bx+C}{x^2+1} &= \frac{4x}{(x-1)(x^2+1)}\\ A(x^2+1) + (Bx+C)(x-1) &= 4x\\ Ax^2 + A + Bx^2 - Bx + Cx - C &= 4x\\ (A+B)x^2 + (C-B)x + (A-C) &= 0x^2 + 4x + 0\end{aligned}$$

$\implies A = 2, B = -2, C = 2$. So we get

$$\begin{aligned}\int \frac{4x}{(x-1)(x^2+1)}\,dx &= \int \frac{2}{x-1} + \frac{-2x+2}{x^2+1}\,dx\\ &= \int 2\frac{1}{x-1} - \frac{2x}{x^2+1} + 2\frac{1}{x^2+1}\,dx\\ &= \boxed{\textstyle\int 2\ln|x-1| - \ln(x^2+1) + 2\tan^{-1}(x) + C}\end{aligned}$$

12. (6+4=10 points) Consider the differential equation: $\dfrac{dy}{dx} = \dfrac{2^x}{3y^2}$.

(a) Find the general solution $y(x)$ to the above equation.

Solution.

$$\begin{aligned} \frac{dy}{dx} &= \frac{2^x}{3y^2} \\ 3y^2\ dy &= 2^x\ dx \\ y^3 &= \frac{2^x}{\ln 2} + C \\ y &= \boxed{\sqrt[3]{\frac{2^x}{\ln 2} + C}} \end{aligned}$$

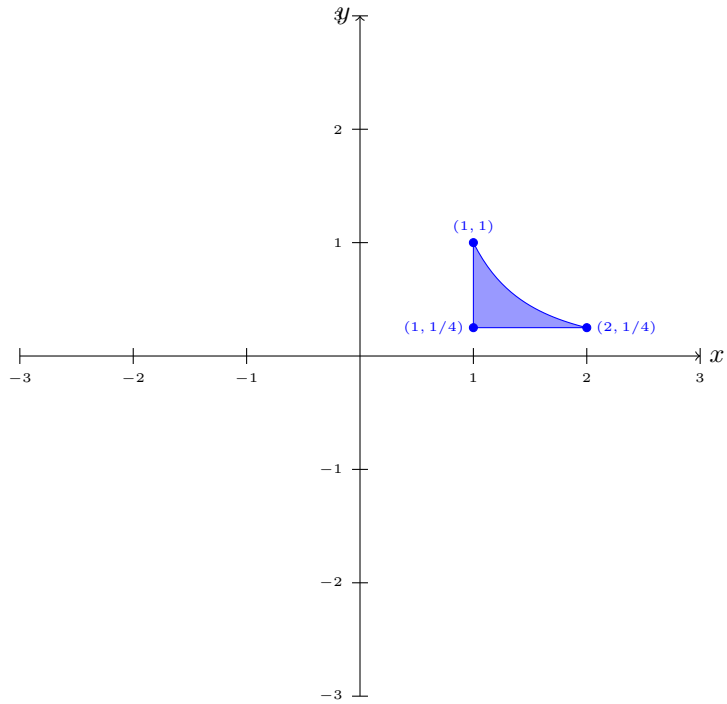
(b) Find the particular solution satisfying the initial condition $y(0) = 0$.

Solution.

$$\begin{aligned} y &= \sqrt[3]{\frac{2^x}{\ln 2} + C} \\ 0 &= \sqrt[3]{\frac{1}{\ln 2} + C} \\ 0 &= \frac{1}{\ln 2} + C \\ -\frac{1}{\ln 2} &= C \\ \implies y &= \boxed{\sqrt[3]{\frac{2^x}{\ln 2} - \frac{1}{\ln 2}}} \\ &= \boxed{\sqrt[3]{\frac{2^x - 1}{\ln 2}}} \end{aligned}$$

13. (4+6=10 points) Consider region in the xy-plane defined by: $x \geq 1$ and $\frac{1}{4} \leq y \leq \frac{1}{x^2}$.

(a) Sketch this region, and mark the coordinates of its corners.



(b) Consider the solid obtained by rotating the region around the axis $x = 1$. Set up an integral to compute the volume of this solid. **You do not need to evaluate the integral.**

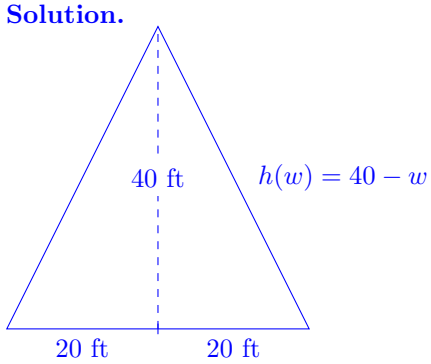
Solution.

$$V = \int_{1/4}^{1} \pi \left[r(y)\right]^2 \, dy$$
$$= \boxed{\pi \int_{1/4}^{1} \left[\frac{1}{\sqrt{y}} - 1\right]^2 \, dy}$$

14. (4+6=10 points) A pyramid has as its base a 40 ft $\times$ 40 ft square at ground level. Its peak is 40 ft directly above the center of the base square.

(a) What is the area of a horizontal square slice h feet above the ground

Solution.



$$A = w^2$$
$$A = (40 - h)^2$$

(b) The pyramid is built of stone weighing 150 pounds per cubic foot. Set up an integral to compute the work needed to lift the stones from ground level to their heights in the pyramid. **You do not need to evaluate the integral.**

Solution.

$$\begin{aligned} W &= \int_0^{40} F(h)\ dh \\ &= \int_0^{40} 150 d(h) A(h)\ dh \\ &= \boxed{\int_0^{40} 150 h(40-h)^2\ dh} \end{aligned}$$