Multiple Choice. Circle the best answer. No work needed. No partial credit available.

1. (5 points) Parametrize of the part of the plane $3x + 2y + z = 10$ that lies above the ellipse $x^2 + 4y^2 = 4$.

   A. $\mathbf{r}(s,t) = \langle 2s\cos t, s\sin t, 10 - 6s\cos t - 2s\sin t\rangle$ with $t \in [0, 2\pi]$ and $s \in [0,1]$.

   B. $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, 10 - 3s\cos t - 2s\sin t\rangle$ with $t \in [0, 2\pi]$ and $s \in [0,2]$.

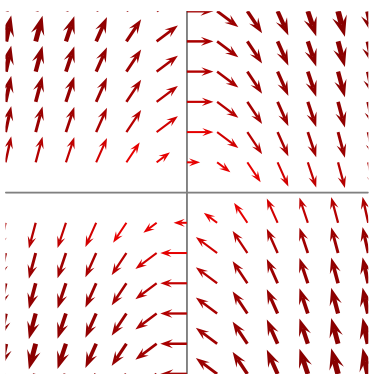
   C. $\mathbf{r}(s,t) = \langle s, t, 10 - 3s - 2t\rangle$, with $s \in [-2,2]$, and $t \in [-1,1]$.

   D. $\mathbf{r}(s,t) = \langle s, t, -3s - 2t\rangle$, with $s \in [-2,2]$, and $t \in [-1,1]$.

   E. None of the above.

2. (5 points) Which of the following vector field plots could be $\mathbf{F} = y\,\mathbf{i} - xy\,\mathbf{j}$?

D.



Extra Work Space.

Fill in the Blanks. No work needed. No partial credit available.

3. Let $\mathbf{F} = (5x^3z)\,\mathbf{i} + (xyz)\,\mathbf{j} + (-6yz^2)\,\mathbf{k}$.

   (a) (5 points) $\operatorname{div}\mathbf{F} =$ $15x^2z + xz - 12yz$.

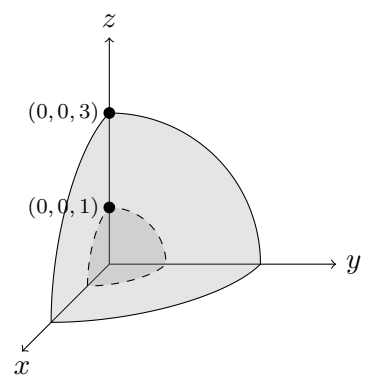
   (b) (5 points) $\operatorname{curl}\mathbf{F} =$ $-\mathbf{i}\left(xy + 6z^2\right) + \mathbf{j}\left(5x^3\right) + \mathbf{k}\,(yz)$.

4. (5 points) If a smooth surface S is given parametrically as $\mathbf{r}(u,v)$ with $(u,v) \in D$, then its surface area is given by the formula:

$$A(S) = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA \quad .$$

5. (10 points) The volume in the first octant between the spheres shown to the right is given by:

$$\int_0^{\theta_2}\int_0^{\phi_2}\int_{\rho_1}^{\rho_2} \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$



where

$\rho_1 =$ 1

$\rho_2 =$ 3

$\phi_2 =$ $\pi/2$

$\theta_2 =$ $\pi/2$

Extra Work Space.

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

6. (12 points) Let $f(x, y, z) = x \sin y + e^{xy} - \ln z$.

(a) Find ∇f at $P_0(3, 0, 2)$.

Solution:

$$\begin{aligned} f_x &= \sin y + ye^{xy} \Longrightarrow f_x(P_0) = 0 \\ f_y &= x \cos y + xe^{xy} \Longrightarrow f_y(P_0) = 6 \\ f_z &= \frac{-1}{z} \Longrightarrow f_z(P_0) = \left(\frac{-1}{2}\right) \end{aligned}$$

Thus

$$(\nabla f)_{P_0} = 0\mathbf{i} + 6\,\mathbf{j} + \left(\frac{-1}{2}\right)\mathbf{k}$$

(b) Find the derivative of f at P_0 in the direction of $\mathbf{A} = \langle 1, 2, -2 \rangle$.

Solution:

Let $\mathbf{u} = \dfrac{\mathbf{A}}{|\mathbf{A}|} = \dfrac{1}{3}\left(\langle 1, 2, -2 \rangle\right)$. Then

$$\begin{aligned} D_{\mathbf{u}} f &= \nabla f \cdot \mathbf{u} \\ &= \frac{1}{3}\left(0\,\mathbf{i} + 6\,\mathbf{j} + \left(\frac{-1}{2}\right)\mathbf{k}\right) \cdot \left(\langle 1, 2, -2 \rangle\right) \\ &= \frac{13}{3} \end{aligned}$$

7. (12 points) Let $f(x, y) = 4x^2 + y^3 - 6xy$. Find and classify each critical point of f as a local maximum, a local minimum, or a saddle point.

Solution:

i. Find the critical points. Notice that $f_x = 8x - 6y$ and $f_y = 3y^2 - 6x$. So f has a critical points at $P(0, 0)$ and $Q(9/8, 3/2)$.

ii. Now let $D(x, y) = f_{xx}\, f_{yy} - f_{xy}^2 = 48y - 36$. Notice that $f_{xx} = 8 > 0$ and that

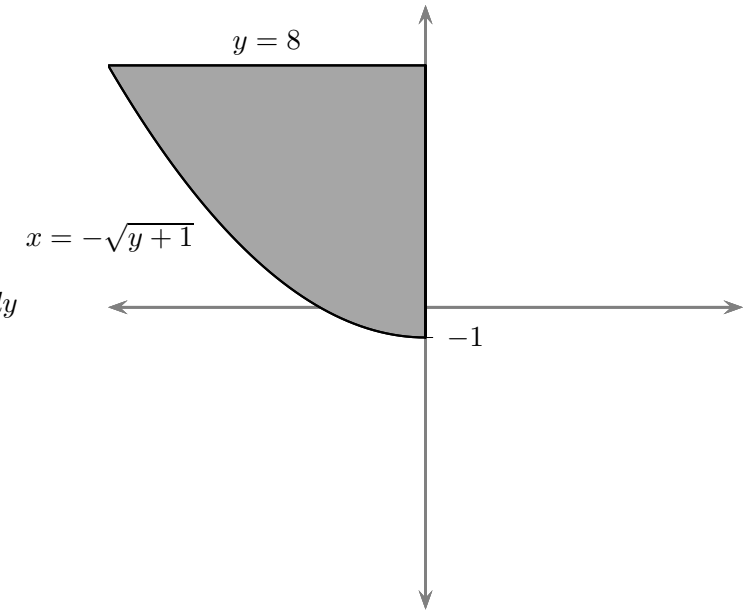
$$D(0, 0) = -36 < 0 \quad \text{and} \quad D(9/8, 3/2) = 36 > 0$$

It follows that f has a local minimum at Q and a saddle point at P.

8. (12 points) Sketch the region of integration for the integral below and write an equivalent integral with the order of integration reversed. **Do not evaluate the integral.**

$$\int_{-3}^{0}\int_{x^2-1}^{8}(x+3y^2)\,dy\,dx$$

Solution:



$$= \int_{-1}^{8}\int_{-\sqrt{y+1}}^{0}(x+3y^2)\,dx\,dy$$

9. (12 points) Evaluate the integral below.

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\cos\left(4x^2+4y^2\right)\,dy\,dx$$

Solution:

We switch to polar coordinates

$$= \int_0^{\pi}\int_0^3 \left(\cos 4r^2\right) r\,dr\,d\theta$$

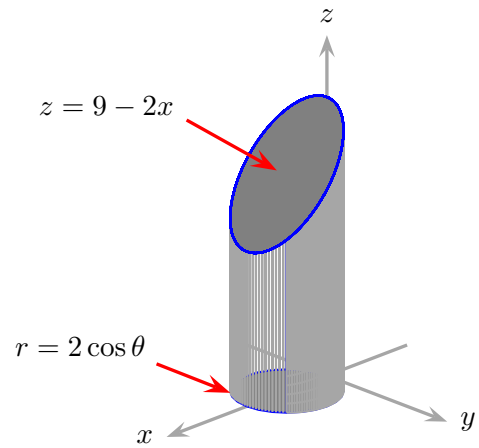
$$= \pi\int_0^3 \left(\cos 4r^2\right) r\,dr$$

$$= \frac{\pi}{8}\int_0^{36}\cos u\,du = \frac{\pi}{8}\sin u\,\Big|_0^{36} = \frac{\pi}{8}\sin 36$$

10. (12 points) Set up but **do not evaluate** the iterated integral for computing the volume of a region D if D is the right circular cylinder whose base is the disk $r = 2\cos\theta$ (in the xy-plane) and whose top lies in the plane $z = 9 - 2x$.

Solution:

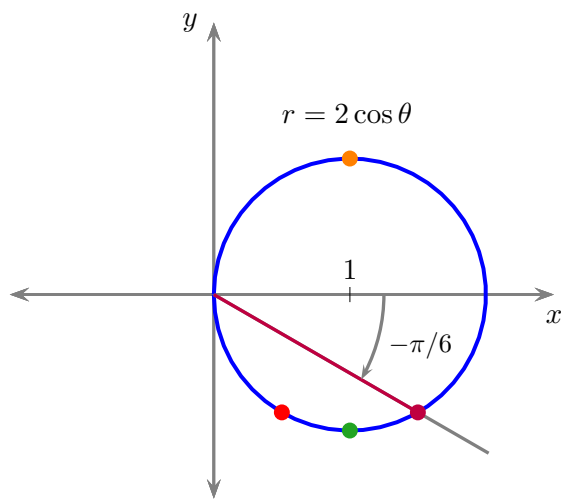
$$\iiint_D dV = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^{9-2x} r\,dz\,dr\,d\theta$$



Of course, $9 - 2x = 9 - 2r\cos\theta$.

The base of the cylinder is shown in the sketch below. We plot a few points to justify the limits of integration for θ in problems 10 and 11. From Math 133...

θ	$r = 2\cos\theta$	(x, y)	Plot Symbol
$-\pi/2$	0	$(0, 0)$	
$-\pi/3$	$1/2$	$(1/2, -\sqrt{3}/2)$	●
$-\pi/4$	$\sqrt{2}$	$(1, -1)$	●
$-\pi/6$	$\sqrt{3}$	$(3/2, -\sqrt{3}/2)$	●
0	2	$(2, 0)$	
$\pi/6$	$\sqrt{3}$	$(3/2, \sqrt{3}/2)$	
$\pi/4$	$\sqrt{2}$	$(1, 1)$	●
$\pi/3$	$1/2$	$(1/2, \sqrt{3}/2)$	
$\pi/2$	0	$(0, 0)$	



For example, the length of the purple line segment is $\sqrt{3}$.

11. (10 points) Find the area of the top of the cylinder in Problem 10.

Solution:

This is straightforward. Let R be the circle $r = 2\cos\theta$ and its interior. Notice that the area of R is π. Now

$$\begin{aligned}
\text{SA} &= \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\
&= \iint_R \sqrt{1 + (-2)^2}\, dA \\
&= \sqrt{5} \underbrace{\iint_R dA}_{\text{area of } R} = \sqrt{5}\pi
\end{aligned}$$

For those that remain unconvinced, we evaluate the double integral (using polar coordinates).

$$\begin{aligned}
\text{SA} &= \sqrt{5} \iint_R dA \\
&= \sqrt{5} \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r\, dr\, d\theta \\
&= \frac{\sqrt{5}}{2} \int_{-\pi/2}^{\pi/2} r^2 \Big|_0^{2\cos\theta} d\theta \\
&= 2\sqrt{5} \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta \\
&= \sqrt{5} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta \\
&= \sqrt{5} \left(\theta + \frac{\sin 2\theta}{2}\right) \Big|_{-\pi/2}^{\pi/2} = \sqrt{5}\pi
\end{aligned}$$

as we observed above.

12. (10 points) Find the work done by the force $\mathbf{F} = \langle 10y, 4x \rangle$ along the straight line segment from $(3, 5)$ to $(2, 0)$.

Solution:

Let C be the indicated line segment. Now let

$$\mathbf{r}(t) = (3 - t)\,\mathbf{i} + (5 - 5t)\,\mathbf{j}, \qquad 0 \le t \le 1$$

Then

$$d\mathbf{r} = ((-1)\,\mathbf{i} + (-5)\,\mathbf{j})\ dt \text{ and } \mathbf{F} \cdot d\mathbf{r} = -10(11 - 7t)\,dt$$

it follows that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= -10 \int_0^1 (11 - 7t)\,dt \\ &= -5(22t - 7t^2)\Big|_0^1 \\ &= -75\end{aligned}$$

13. (a) (6 points) Find a function f so that $\nabla f = y^3\,\mathbf{i} + (3xy^2 - 5z^2)\,\mathbf{j} + (-10yz)\,\mathbf{k}$.

Solution:

Let

$$f(x, y, z) = xy^3 - 5yz^2$$

then

$$\nabla f = y^3\,\mathbf{i} + (3xy^2 - 5z^2)\,\mathbf{j} + (-10yz)\,\mathbf{k}$$

(b) (5 points) Evaluate the integral below.

$$\int_C y^3\,dx + (3xy^2 - 5z^2)\,dy + (-10yz)\,dz$$

Here C is any path from $(1, 2, 0)$ to $(2, 1, -1)$.

Solution:

Observe that

$$df = y^3\,dx + (3xy^2 - 5z^2)\,dy + (-10yz)\,dz$$

It follows that

$$\begin{aligned}\int_{(1,2,0)}^{(2,1,-1)} y^3\,dx + (3xy^2 - 5z^2)\,dy + (-10yz)\,dz &= \int_{(1,2,0)}^{(2,1,-1)} df \\ &= f(x, y, z)\Big|_{(1,2,0)}^{(2,1,-1)} \\ &= -3 - 8\end{aligned}$$

14. (12 points) Let $E = \{(x, y, z) \,|\, 0 \le x \le z,\ 1 \le y \le 5,\ y \le z \le 5\}$. Rewrite the triple integral below as an iterated integral and evaluate.

$$\iiint_E dV$$

Solution:

$$\begin{aligned}
\iiint_E dV &= \int_1^5 \int_y^5 \int_0^z dx\, dz\, dy \\
&= \int_1^5 \int_y^5 z\, dz\, dy \\
&= \frac{1}{2} \int_1^5 (25 - y^2) dy \\
&= \frac{1}{2} \left(25y - \frac{y^3}{3} \right) \Big|_1^5 \\
&= \frac{88}{3}
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\iiint_E dV &= \int_1^5 \int_0^z \int_1^z dy\, dx\, dz \\
&= \int_1^5 (z-1) \int_0^z dx\, dz \\
&= \int_1^5 (z-1) z\, dz \\
&= \left(\frac{z^3}{3} - \frac{z^2}{2} \right) \Big|_1^5 \\
&= \frac{88}{3}
\end{aligned}$$

15. (12 points) Find the work done by the force $\mathbf{F} = 6xy\,\mathbf{i} + \left(3x^2 + 2x\right)\mathbf{j}$ when moving a particle around the circle $x^2 + y^2 = 16$ starting and ending at $(4, 0)$ traveling in the counterclockwise direction.

Solution:

Let C be the circle of radius 4 centered at the origin.

Method 1: Direct calculation.

Now C is given by the parametric equation

$$\begin{aligned} \mathbf{r}(t) &= (4\cos t)\,\mathbf{i} + (4\sin t)\,\mathbf{j}, \quad 0 \le t \le 2\pi \\ d\mathbf{r} &= \left((-4\sin t)\,\mathbf{i} + (4\cos t)\,\mathbf{j}\right) dt \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{r} &= (6xy)\,(-4\sin t)\,dt + \left(3x^2 + 2x\right)(4\cos t)\,dt \\ &= \vdots \\ &= 16\left[-24\sin^2 t\cos t + 12\cos^3 t + 2\cos^2 t\right] dt \end{aligned}$$

It follows that the work done is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= 16\int_0^{2\pi} \left[-24\sin^2 t\cos t + 12\cos^3 t + 2\cos^2 t\right] dt \\ &= \vdots \\ &= 16\int_0^{2\pi} (1 + \cos 2t)dt \\ &= 32\pi \end{aligned}$$

Method 2: Green's Theorem

Let R be the interior of the circle C. Then the area of R is 16π and by Green's Theorem

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \left(\frac{\partial\left(3x^2 + 2x\right)}{\partial x} - \frac{\partial\,(6xy)}{\partial y}\right) dA \\ &= \iint_R (2)\,dA \\ &= 2 \times \text{Area of } R = 32\pi \end{aligned}$$

as we saw above.

Clearly, this is the easier of the two calculations.