Multiple Choice. Circle the best answer. No work needed. No partial credit available.

1. (5 points) Parametrize of the part of the plane $3x + 2y + z = 10$ that lies above the ellipse $x^2 + 4y^2 = 4$.
   A. $\mathbf{r}(s, t) = \langle 2s \cos t, s \sin t, 10 - 6s \cos t - 2s \sin t \rangle$ with $t \in [0, 2\pi]$ and $s \in [0, 1]$.
   B. $\mathbf{r}(s, t) = \langle s \cos t, s \sin t, 10 - 3s \cos t - 2s \sin t \rangle$ with $t \in [0, 2\pi]$ and $s \in [0, 2]$.
   C. $\mathbf{r}(s, t) = \langle s, t, 10 - 3s - 2t \rangle$, with $s \in [-2, 2]$, and $t \in [-1, 1]$.
   D. $\mathbf{r}(s, t) = \langle s, t, -3s - 2t \rangle$, with $s \in [-2, 2]$, and $t \in [-1, 1]$.
   E. None of the above.

2. (5 points) Which of the following vector field plots could be $\mathbf{F} = y \mathbf{i} - xy \mathbf{j}$?
   
   D. ![Vector Field Plot]

Extra Work Space.
3. Let \( \mathbf{F} = (5x^3 z)\mathbf{i} + (xyz)\mathbf{j} + (-6yz^2)\mathbf{k} \).
   
   (a) (5 points) \( \text{div} \mathbf{F} = 15x^2 z + xz - 12yz \).
   
   (b) (5 points) \( \text{curl} \mathbf{F} = - \mathbf{i}(xy + 6z^2) + \mathbf{j}(5x^3) + \mathbf{k}(yz) \).

4. (5 points) If a smooth surface \( S \) is given parametrically as \( \mathbf{r}(u, v) \) with \( (u, v) \in D \), then its surface area is given by the formula:
   \[
   A(S) = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA.
   \]

5. (10 points) The volume in the first octant between the spheres shown to the right is given by:
   \[
   \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{\rho=1}^{3} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
   \]
   where
   \[
   \rho_1 = 1
   \]
   \[
   \rho_2 = 3
   \]
   \[
   \phi_2 = \pi/2
   \]
   \[
   \theta_2 = \pi/2
   \]
6. (12 points) Let \( f(x, y, z) = x \sin y + e^{xy} - \ln z \).

(a) Find \( \nabla f \) at \( P_0(3, 0, 2) \).

Solution:

\[
\begin{align*}
    f_x &= \sin y + ye^{xy} \implies f_x(P_0) = 0 \\
    f_y &= x \cos y + xe^{xy} \implies f_y(P_0) = 6 \\
    f_z &= -\frac{1}{z} \implies f_z(P_0) = \left( \frac{-1}{2} \right)
\end{align*}
\]

Thus

\[
(\nabla f)_{P_0} = 0i + 6j + \left( \frac{-1}{2} \right)k
\]

(b) Find the derivative of \( f \) at \( P_0 \) in the direction of \( \mathbf{A} = \langle 1, 2, -2 \rangle \).

Solution:

Let \( \mathbf{u} = \frac{\mathbf{A}}{||\mathbf{A}||} = \frac{1}{3} \langle 1, 2, -2 \rangle \). Then

\[
D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}
\]

\[
= \frac{1}{3} \left( 0 \mathbf{i} + 6 \mathbf{j} + \left( \frac{-1}{2} \right) \mathbf{k} \right) \cdot \langle 1, 2, -2 \rangle
\]

\[
= \frac{13}{3}
\]
7. (12 points) Let \( f(x, y) = 4x^2 + y^3 - 6xy \). Find and classify each critical point of \( f \) as a local maximum, a local minimum, or a saddle point.

**Solution:**

i. Find the critical points. Notice that \( f_x = 8x - 6y \) and \( f_y = 3y^2 - 6x \). So \( f \) has a critical points at \( P(0, 0) \) and \( Q(9/8, 3/2) \).

ii. Now let \( D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 48y - 36 \). Notice that \( f_{xx} = 8 > 0 \) and that

\[
D(0, 0) = -36 < 0 \quad \text{and} \quad D(9/8, 3/2) = 36 > 0
\]

It follows that \( f \) has a local minimum at \( Q \) and a saddle point at \( P \).
8. (12 points) Sketch the region of integration for the integral below and write an equivalent integral with the order of integration reversed. **Do not evaluate the integral.**

\[
\int_{3}^{0} \int_{x^2-1}^{8} (x + 3y^2) \, dy \, dx
\]

Solution:

\[
= \int_{-1}^{8} \int_{-\sqrt{y+1}}^{0} (x + 3y^2) \, dx \, dy
\]

9. (12 points) Evaluate the integral below.

\[
\int_{-3}^{3} \int_{\sqrt{9-x^2}}^{0} \cos (4x^2 + 4y^2) \, dy \, dx
\]

Solution:

We switch to polar coordinates

\[
= \int_{0}^{\pi} \int_{0}^{3} (\cos 4r^2) \, r \, dr \, d\theta
\]

\[
= \pi \int_{0}^{3} (\cos 4r^2) \, r \, dr
\]

\[
= \frac{\pi}{8} \int_{0}^{36} \cos u \, du = \frac{\pi}{8} \sin u \bigg|_{0}^{36} = \frac{\pi}{8} \sin 36
\]
10. (12 points) Set up but do not evaluate the iterated integral for computing the volume of a region $D$ if $D$ is the right circular cylinder whose base is the disk $r = 2\cos \theta$ (in the $xy$-plane) and whose top lies in the plane $z = 9 - 2x$.

Solution:

$$\iiint_D dV = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos \theta} \int_0^{9-2x} r \, dz \, dr \, d\theta$$

Of course, $9 - 2x = 9 - 2r \cos \theta$.

The base of the cylinder is shown in the sketch below. We plot a few points to justify the limits of integration for $\theta$ in problems 10 and 11. From Math 133...

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$r = 2\cos \theta$</th>
<th>$(x, y)$</th>
<th>Plot Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\pi/2$</td>
<td>0</td>
<td>(0,0)</td>
<td></td>
</tr>
<tr>
<td>$-\pi/3$</td>
<td>1/2</td>
<td>$1/2, -\sqrt{3}/2$</td>
<td>●</td>
</tr>
<tr>
<td>$-\pi/4$</td>
<td>$\sqrt{2}$</td>
<td>(1, -1)</td>
<td>●</td>
</tr>
<tr>
<td>$-\pi/6$</td>
<td>$\sqrt{3}$</td>
<td>$(3/2, -\sqrt{3}/2)$</td>
<td>●</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>(2,0)</td>
<td></td>
</tr>
<tr>
<td>$\pi/6$</td>
<td>$\sqrt{3}$</td>
<td>$(3/2, \sqrt{3}/2)$</td>
<td>●</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>$\sqrt{2}$</td>
<td>(1,1)</td>
<td>●</td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>1/2</td>
<td>$1/2, \sqrt{3}/2$</td>
<td>●</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>0</td>
<td>(0,0)</td>
<td></td>
</tr>
</tbody>
</table>

For example, the length of the purple line segment is $\sqrt{3}$. 
11. (10 points) Find the area of the top of the cylinder in Problem 10.

Solution:

This is straightforward. Let $R$ be the circle $r = 2 \cos \theta$ and its interior. Notice that the area of $R$ is $\pi$. Now

$$\text{SA} = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

$$= \iint_R \sqrt{1 + (-2)^2} \, dA$$

$$= \sqrt{5} \int_R dA = \sqrt{5}\pi$$

For those that remain unconvinced, we evaluate the double integral (using polar coordinates).

$$\text{SA} = \sqrt{5} \int_R dA$$

$$= \sqrt{5} \int_{-\pi/2}^{\pi/2} \int_0^{2\cos \theta} r \, dr \, d\theta$$

$$= \sqrt{5} \left. \frac{r^2}{2} \right|_0^{2\cos \theta} \int_{-\pi/2}^{\pi/2} d\theta$$

$$= \sqrt{5} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) \, d\theta$$

$$= \sqrt{5} \left[ \theta + \frac{\sin 2\theta}{2} \right]_{-\pi/2}^{\pi/2} = \sqrt{5}\pi$$

as we observed above.
12. (10 points) Find the work done by the force $\mathbf{F} = \langle 10y, 4x \rangle$ along the straight line segment from $(3, 5)$ to $(2, 0)$.

Solution:

Let $C$ be the indicated line segment. Now let

$$\mathbf{r}(t) = (3-t) \mathbf{i} + (5-5t) \mathbf{j}, \quad 0 \leq t \leq 1$$

Then

$$d\mathbf{r} = ((-1) \mathbf{i} + (-5) \mathbf{j}) \, dt$$

and $\mathbf{F} \cdot d\mathbf{r} = -10(11-7t) \, dt$

it follows that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -10 \int_0^1 (11 - 7t) \, dt$$

$$= -5(22t - 7t^2) \bigg|_0^1$$

$$= -75$$
13. (a) (6 points) Find a function \( f \) so that \( \nabla f = y^3 \mathbf{i} + (3xy^2 - 5z^2) \mathbf{j} + (-10yz) \mathbf{k} \).

**Solution:**

Let

\[ f(x, y, z) = xy^3 - 5yz^2 \]

then

\[ \nabla f = y^3 \mathbf{i} + (3xy^2 - 5z^2) \mathbf{j} + (-10yz) \mathbf{k} \]

(b) (5 points) Evaluate the integral below.

\[ \int_C y^3 \, dx + (3xy^2 - 5z^2) \, dy + (-10yz) \, dz \]

Here \( C \) is any path from \((1,2,0)\) to \((2,1,-1)\).

**Solution:**

Observe that

\[ df = y^3 \, dx + (3xy^2 - 5z^2) \, dy + (-10yz) \, dz \]

It follows that

\[ \int_{(1,2,0)}^{(2,1,-1)} y^3 \, dx + (3xy^2 - 5z^2) \, dy + (-10yz) \, dz \]

\[ = f(x, y, z) \bigg|_{(1,2,0)}^{(2,1,-1)} \]

\[ = -3 - 8 \]
14. (12 points) Let \( E = \{(x, y, z) \mid 0 \leq x \leq z, 1 \leq y \leq 5, y \leq z \leq 5\} \). Rewrite the triple integral below as an iterated integral and evaluate.

\[
\iiint_E dV
\]

Solution:

\[
\iiint_E dV = \int_1^5 \int_y^5 \int_0^z dx \, dz \, dy
\]
\[
= \int_1^5 \int_y^5 z \, dz \, dy
\]
\[
= \frac{1}{2} \int_1^5 (25 - y^2) \, dy
\]
\[
= \frac{1}{2} \left( 25y - \frac{y^3}{3} \right) \bigg|_1^5
\]
\[
= \frac{88}{3}
\]

Equivalently,

\[
\iiint_E dV = \int_1^5 \int_0^z \int_y^z dy \, dx \, dz
\]
\[
= \int_1^5 (z - 1) \int_0^z dx \, dz
\]
\[
= \int_1^5 (z - 1)z \, dz
\]
\[
= \left( \frac{z^3}{3} - \frac{z^2}{2} \right) \bigg|_1^5
\]
\[
= \frac{88}{3}
\]
15. (12 points) Find the work done by the force \( \mathbf{F} = 6xy \mathbf{i} + (3x^2 + 2x) \mathbf{j} \) when moving a particle around the circle \( x^2 + y^2 = 16 \) starting and ending at \((4,0)\) traveling in the counterclockwise direction.

Solution:

Let \( C \) be the circle of radius 4 centered at the origin.

**Method 1:** Direct calculation.

Now \( C \) is given by the parametric equation

\[
\mathbf{r}(t) = (4 \cos t) \mathbf{i} + (4 \sin t) \mathbf{j}, \quad 0 \leq t \leq 2\pi
\]

\[
d\mathbf{r} = ((-4 \sin t) \mathbf{i} + (4 \cos t) \mathbf{j}) dt
\]

Hence

\[
\mathbf{F} \cdot d\mathbf{r} = (6xy)(-4 \sin t) dt + (3x^2 + 2x)(4 \cos t) dt
\]

\[
= \;
\]

\[
= 16 \left[ -24 \sin^2 t \cos t + 12 \cos^3 t + 2 \cos^2 t \right] dt
\]

It follows that the work done is

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = 16 \int_0^{2\pi} \left[ -24 \sin^2 t \cos t + 12 \cos^3 t + 2 \cos^2 t \right] dt
\]

\[
= \;
\]

\[
= 16 \int_0^{2\pi} (1 + \cos 2t) dt
\]

\[
= 32\pi
\]

**Method 2:** Green’s Theorem

Let \( R \) be the interior of the circle \( C \). Then the area of \( R \) is \( 16\pi \) and by Green’s Theorem

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial (3x^2 + 2x)}{\partial x} - \frac{\partial (6xy)}{\partial y} \right) dA
\]

\[
= \iint_R (2) dA
\]

\[
= 2 \times \text{Area of } R = 32\pi
\]

as we saw above.

Clearly, this is the easier of the two calculations.