Chapter 3
Applications of Differentiation

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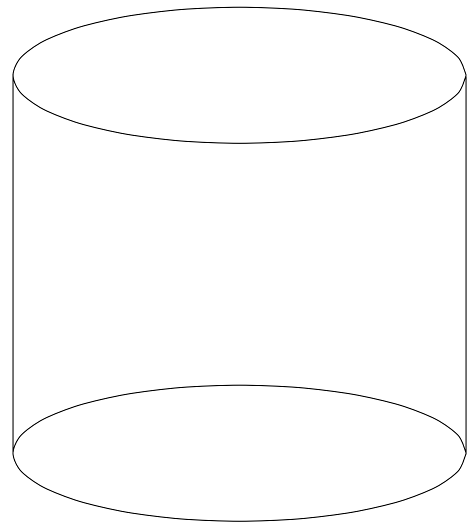
Motivation to Chapter 3

It seems in the real world everyone likes minimizing and maximizing. Minimizing costs, maximizing profits, minimizing spread of disease, maximizing algorithm efficiency, and so so much more. And so in Chapter 3 a lot of our time is spent on this idea of maximizing and minimizing (optimization for short). Here's a nice example to wet your appetite.

Example: Suppose you have a fixed amount of metal to build a soup can.

What radius and height will maximize the volume the can will hold?



Use `https://www.desmos.com/calculator/yu5kis7gzb` to help solve this problem.

Another key topic we will discuss is how a function's derivative and second derivative impacts the shape of a function's graph. Smaller topics we will visit along the way include:

- Using the Mean Value Theorem to guarantee that the derivative takes on a specific value.
- Using Newton's Method to show two graphs intersect.
- Calculating anti-derivatives (which will lead nicely into CH4).

1 Maximum and Minimum Values

1.1 VIDEO - Absolute Mins and Maxs and Why They Don't Always Exist

Objective(s):

- Define maximums and minimums and be able to visualize them graphically.
- Comprehend why absolute mins and maxs don't always exist.
- Gain exposure to the Extreme Value Theorem.

Definition(s) 1.1. Let c be a number in the domain D of a function f. Then $f(c)$ is the

- __________________________________ value of f on D if $f(c) \geq f(x)$ for all x in D.
- __________________________________ value of f on D if $f(c) \leq f(x)$ for all x in D.
- ____________________ value of f on D if $f(c) \geq f(x)$ for all x near c.
- ____________________ value of f on D if $f(c) \leq f(x)$ for all x near c.

Maximums and minimums are often referred to as **extreme values**.

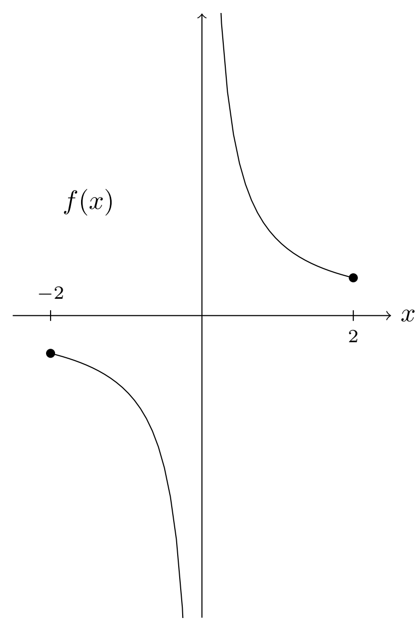
Pictures:

Remark 1.2. Maximums and minimums must be finite real numbers

Remark 1.3. The book uses "**near** c" to mean technically that the statement is true in some **open** interval containing c. So technically endpoints cannot be local mins/maxes... Sometimes these definitions can make you (and me) crazy!

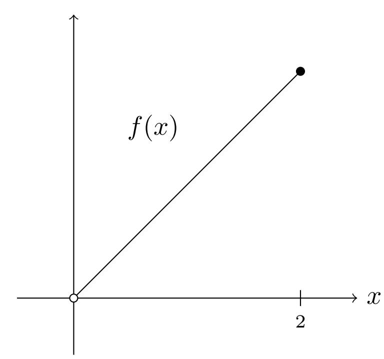
Example 1.4. Consider the graph of $f(x) = \dfrac{1}{x}$ on the interval $[-2, 2]$.

Explain why there is no absolute minimum for $f(x)$ on this interval.



Example 1.5. Consider the graph of $f(x) = x$ on the interval $(0, 2]$.

Explain why there is no absolute minimum for $f(x)$ on this interval.



Theorem 1.6 (Extreme Value Theorem (EVT)). If f is ____________________ on a ___________ interval $[a, b]$, then f attains an __________________________________ value and an _________________________________ in $[a, b]$.

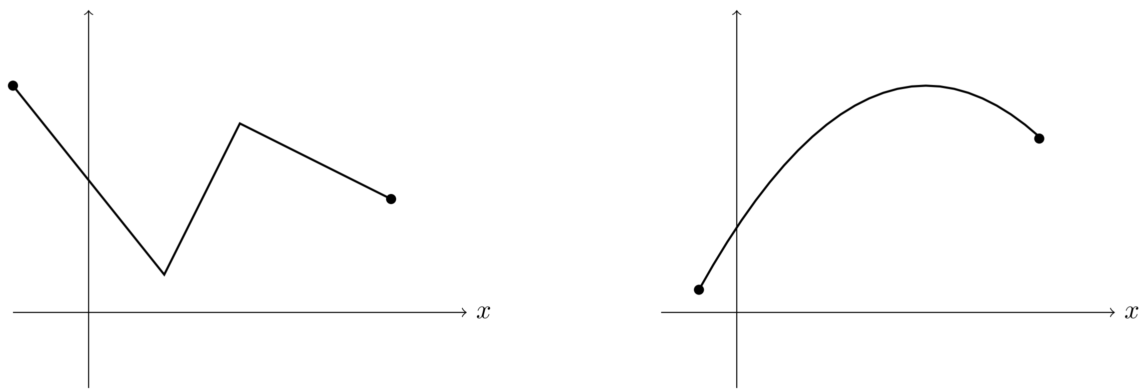
Remark 1.7. This theorem is nice and all but it only guarantees that the maximum and minimum exist... it doesn't tell us how to find them. We will go over this strategy next time!

1.2 VIDEO - The Extreme Value Theorem (Finding Absolute Minima and Maxima)

Objective(s):

- Explore a strategy for finding absolute mins and maxes.
- Practice finding minimums and maximums!

We have seen that closed intervals are good. Let's explore where minimums and maximums occur for continuous functions on closed intervals so we can develop general strategies.



Remark 1.8. Absolute extremum seem to appear possibly at ________________, or when the derivative is _______________________.

Definition(s) 1.9. A ____________________________ of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 1.10. Find the critical numbers for the function $f(x) = 2x^3 - 3x^2 - 12x + 5$.

Theorem 1.11. To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

(a) Find the values of f at the __________________________ of f in (a, b).

(b) Find the values of f at ______________________ (a and b).

(c) The largest of the values from above is the absolute ________________ value;

the smallest is the absolute ________________ value.

Example 1.12. Find the absolute maximum an minimum values of $f(x) = x^3 - 12x + 1$ on the interval $[1, 4]$

Example 1.13. Find the absolute maximum an minimum values of $g(t) = t - 3t^{2/3}$ on the interval $[-1, 27]$

2 The Mean Value Theorem

2.1 VIDEO - Statements and Meanings

Objective(s):

- State the Mean Value Theorem and draw pictures to help us understand its meaning.
- Identify points on the correct interval that satisfy the Mean Value Theorem.

Theorem 2.1 (Rolle's Theorem)**.** Let $f(x)$ be a function which satisfies the following three properties:

(1) $f(x)$ is ________________________ the interval $[a, b]$

(2) $f(x)$ is ____________________________ (a, b)

(3)

Then there is a number c in (a, b) such that _______________.

Remark 2.2. The conclusion of **Rolle's Theorem** says that if the function values agree at the endpoints, then there is a place in between where the tangent line is horizontal.

Picture:

Theorem 2.3 (**Mean Value Theorem** (MVT))**.** Let $f(x)$ be a function which satisfies the following two properties:

(1) $f(x)$ is continuous on the interval $[a, b]$

(2) $f(x)$ is differentiable on (a, b)

Then there is a number c in (a, b) such that

$$f'(c) =$$

Remark 2.4. The conclusion of **The Mean Value Theorem** says that there is a place in the interval where the tangent line is ______________ to the ____________ line between the endpoints.

Remark 2.5. Notice that **Rolle's Theorem** and **The Mean Value Theorem** tell you that "there exists" a number c with certain properties, but neither theorem tells you what that the value of c is, or how to find it.

Picture:

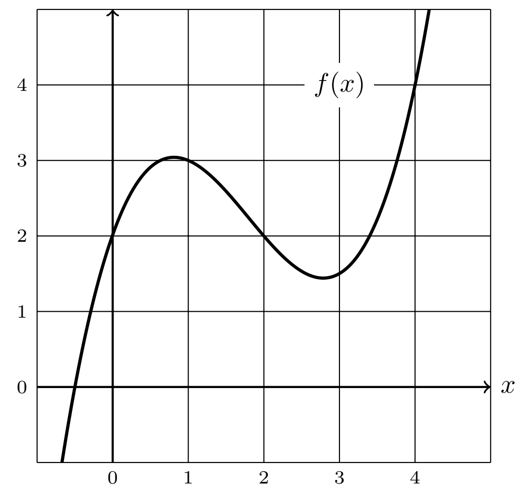
2.2 VIDEO - Practice Time

Objective(s):

- Apply the Mean Value Theorem to functions given graphically.
- Apply the Mean Value Theorem to functions given via equations.
- Re-emphasize when the Mean Value Theorem does not apply.

Example 2.6. Consider the function $f(x)$ given by the graph on the right.

Find a value c that satisfies the conclusion of the MVT on the interval $[1, 4]$.



Example 2.7. Consider the function $f(x) = x^2 + 3x + 5$. Can the MVT be applied to f on the interval $[0, 1]$?

(i) If yes, find a c value that satisfies the conclusion of the MVT.

(ii) If no, explain why not.

Example 2.8. Consider the function $f(x) = \frac{1}{x}$. Can the MVT be applied to f on the interval $[1, 3]$?

(i) If yes, find a c value that satisfies the conclusion of the MVT.

(ii) If no, explain why not.

Example 2.9. Consider the function $f(x) = |x|$. Can the MVT be applied to f on the interval $[-2, 2]$?

(i) If yes, find a c value that satisfies the conclusion of the MVT.

(ii) If no, explain why not.

2.3 VIDEO - A Strange Consequence

Objective(s):

- Understand an interesting result of the MVT and why we care.
- Solve a few problems related to this idea.

Corollary 2.10.

If ________________ for all x in an interval (a, b), then $f(x)$ must be constant on (a, b).

Remark 2.11. Why does it make sense?

Remark 2.12. Why does it matter?

Corollary 2.13.

If ______________________ for all x in an interval (a, b), then $f(x) = g(x) + c$ for some constant c.

Example 2.14. (a) Find a function $f(x)$ that satisfies $f'(x) = 2x$.

(b) Find a different function $f(x)$ that satisfies $f'(x) = 2x$.

3 Derivatives and Graphs

3.1 OPTIONAL VIDEO - Inequality Review

Objective(s):

- Solve a few inequality problems to get the juices flowing.

We are about to enter a part of calculus in which we will be solving lots of inequalities and so it is important that you remember how! Here are a few problems to help jog your memory.

Example 3.1. Solve the inequality $x^2 - 2x - 3 > 0$

Example 3.2. Solve the inequality $\dfrac{x+5}{x-7} < 0$

Example 3.3. Solve the inequality $1 - \dfrac{3}{x-2} > 0$

3.2 VIDEO - Increasing, Decreasing, and Concavity

Objective(s):

- Utilize the derivative to determine when a function is increasing or decreasing.
- Examine the second derivative to determine when a function is concave up or down.

We are now embarking on a journey to be able to do fairly detailed sketches of functions using their derivatives.

Theorem 3.4.

(a) If $f'(x) > 0$ on (a, b), then $f(x)$ is __________________ on (a, b).

(b) If $f'(x) < 0$ on (a, b), then $f(x)$ is __________________ on (a, b).

Example 3.5. Find where $f(x) = x^2 + 4x + 5$ is increasing.

Definition(s) 3.6.

- If the graph of f lies above all of its tangents on an interval I, then it is called ______________________ on I.
- If the graph of f lies below all of its tangents on I, it is called ____________________________ on I.

Picture:

Theorem 3.7 (Concavity Test)**.**

(a) If __________________ for all x in I, then the graph of f is concave upward on I.

(b) If __________________ for all x in I, then the graph of f is concave downward on I.

Example 3.8. Find where $f(x) = x^2 + 4x + 5$ is concave up.

3.3 VIDEO - Local Mins and Maxes

Objective(s):

- Find local mins and maxes of a function given to us via an equation.

In section 1 of chapter 3 we defined **critical numbers** and saw that these were places where minimums and maximums can occur. In this video we hope to use the first derivative to help classify critical numbers. First let's notice something ...

Example 3.9. Sketch as function that is increasing on $(-\infty, -1) \cup (3, \infty)$ and is decreasing on $(-1, 3)$.

Theorem 3.10 (First Derivative Test)**.**

Suppose that $f(x)$ is a function and that c is a ________________________ of $f(x)$.

(a) If $f'(x)$ changes from ______________________________ at $x = c$,

then $f(x)$ has a local ________________ at $x = c$.

(b) If $f'(x)$ changes from ______________________________ at $x = c$,

then $f(x)$ has a local _______________ at $x = c$.

(c) If $f'(x)$ does not ___________________ at $x = c$,

then $f(x)$ has ____________ a local maximum nor a local minimum at $x = c$.

Pictures:

Example 3.11. For the following functions, find the intervals on which it is increasing and decreasing, and find where the local maximum and local minimum values occur.

(a) $f(x) = 2x^3 + 3x^2 - 36x$ on the domain $(-\infty, \infty)$

(b) $f(x) = \dfrac{x}{x^2+1}$ on the domain $(-\infty, \infty)$

3.4 VIDEO - Points of Inflection

Objective(s):

- Define inflection points.
- Practice finding inflection points give a function's equation.

Definition(s) 3.12. A point P on a curve $y = f(x)$ is called an ______________________________
if f is continuous there and either

(a) the curve changes from concave upward to concave downward at P.

(b) the curve changes from concave downward to concave upward at P.

Remark 3.13. Because of the above definition we also are interested in where the second derivative is 0 or undefined.

Example 3.14. Suppose $f(x)$ is a continuous function an $f''(x) = (x+1)^2(x+5)$. Find where $f(x)$ is concave up and where it is concave down. Where are the inflection points?

Example 3.15. Find where $f(x) = x^3 - x^2 - 5x + 3$ is concave up and where it is concave down. Where are the inflection points?

4 Limits at Infinity; Horizontal Asymptotes

4.1 VIDEO - A Review of Rational Functions and Algebra

Objective(s):

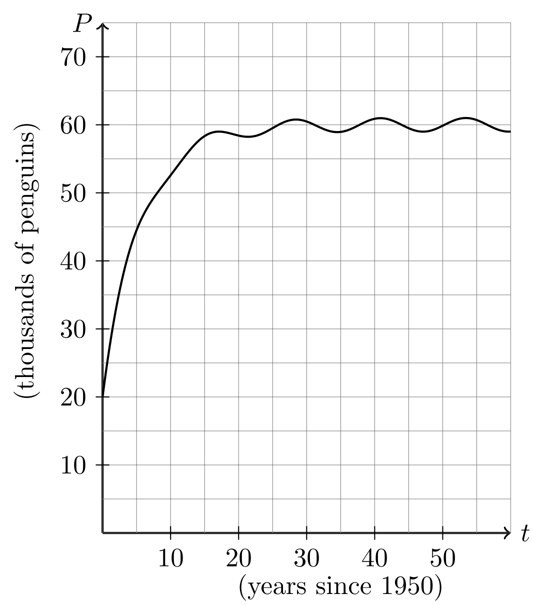
- Review the algebra rules for horizontal asymptotes.

Definition(s) 4.1 (Algebra definition)**.** A horizontal asymptote is a horizontal line that tells you how the function will behave at very large positive values or very small negative values.

Myth: A function ______________________ a horizontal asymptote.

Before we get into Horizontal asymptotes let's recognize that there are many real world applications to horizontal asymptotes so they are indeed worth studying. One such example comes from the world of biology!

Example: For a given region, the **carrying capacity** is the maximum number of individuals of a given species that an area's resources can sustain indefinitely without significantly depleting or degrading those resources. To the right is a graph representing the population of penguins on a particular island. What do you think the carrying capacity for penguins is on this island?



Theorem 4.2. Suppose f and g are polynomials with leading coefficients a and b respectively.

(a) If $\deg(f) > \deg(g)$ then $\frac{f}{g}$ ____.

(b) If $\deg(f) < \deg(g)$ then $\frac{f}{g}$ ____.

(c) If $\deg(f) = \deg(g)$ then $\frac{f}{g}$ ____.

Example 4.3. Graph each of the functions below to verify that the satisfy the conclusions of **Theorem 4.2**.

(a) $y = \frac{x}{1}$

(b) $y = \frac{1}{x}$

(c) $y = \frac{2x}{3x}$

Example 4.4. Use **Theorem 4.2** to find the horizontal asymptotes for

(a) $y = \frac{3x+1}{1-x}$

(b) $y = \frac{x+2x^2+5x^3}{(7+x^2)^2}$

4.2 VIDEO - The Calculus of Horizontal Asymptotes

Objective(s):

- Investigate horizontal asymptotes of a function given algebraically by using limits at infinity.

Definition(s) 4.5. Let f be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that the values of ________ can be made arbitrarily close to ___ by taking x ______________________________.

Definition(s) 4.6. Let f be a function defined on some interval $(-\infty, a)$. Then $\lim_{x\to-\infty} f(x) = L$ means that the values of ________ can be made arbitrarily close to ___ by taking x ______________________________.

Definition(s) 4.7. The line $y = L$ is called a ______________________________ of the curve $y = f(x)$ if either

or

Theorem 4.8. If $r > 0$ then

If $r > 0$ such that x^r is defined for all x, then

Example 4.9. Find the limit or show that it does not exist: $\lim_{x\to\infty} \dfrac{2x-1}{5x+3}$

Remark 4.10 (Calculus Technique for Evaluating Horizontal Asymptotes)**.**

To find the infinite limit divide both the numerator and denominator by the largest power of x in the denominator.

Example 4.11. Evaluate the limit or show that it does not exist: $\lim_{x\to\infty} \frac{\sqrt{x}-x^3}{2x+x^2}$

Example 4.12. Find the horizontal asymptotes for $y = \frac{\sqrt[3]{x}+3x^2}{\sqrt{x}(5+7x^2)}$

5 Curve Sketching

5.1 VIDEO - Slant Asymptotes

Objective(s):

- Define slant asymptotes and review their properties.
- Review polynomial long division.
- Practice finding slant asymptotes.

Vertical asymptotes are when a graph approaches a vertical line. Horizontal asymptotes are when a graph approaches a horizontal line. So what if your graph approaches a line that isn't vertical or horizontal? Welcome to the world of slant asymptotes!

Example 5.1. Use `https://www.desmos.com/calculator/tifbtpclbk` to find the slant asymptote of $y = \dfrac{2x^2 - x}{x+1}$

We see that as x get's close to $\pm\infty$ the graph approaches the straight line. More formally:

Definition(s) 5.2. The function $y = f(x)$ has the slant asymptote $y = mx + b$ if

or if

Okay so what about on homework/quizzes/exams when you don't have a calculator or computer? How do we find slant asymptotes? I'm sorry to say that the answer is... ________________________________.

Example 5.3. Write $\dfrac{219}{12}$ as a mixed fraction by using long division

Example 5.4. Use polynomial long division to simplify $f(x) = \dfrac{2x^2 - x}{x + 1}$

Example 5.5. Use the definition of slant asymptotes to verify that the quotient of your answer in **Example 5.4** is a slant asymptote.

> **Remark 5.6.** After simplifying a rational function using polynomial long division if the quotient is a linear function then it is the ______________________________.

Example 5.7. Find the slope asymptote(s) of $y = \dfrac{x^2 + 3x + 2}{x - 2}$

5.2 VIDEO - Two Curve Sketching Problems

Objective(s):

- Combine your algebra knowledge and Chapter 3 material to sketch curves of functions.

Example 5.8. Consider the function $f(x) = x(x-4)^3$

(a) What is the domain of $f(x)$?

(b) Find the x and y intercepts.

(c) Find all vertical/horizontal/slant asymptotes.

(d) Find where f is increasing and where it is decreasing. Classify all critical points.

(e) Find where f is concave up/down. Identify all inflection points.

(f) Use parts (a)−(e) to sketch $f(x)$.

Example 5.9. Consider the function $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$ and its derivatives $f'(x) = \dfrac{x(x-4)}{(x-2)^2}$ and $f''(x) = \dfrac{8}{(x-2)^3}$

(a) What is the domain of $f(x)$?

(b) Find the x and y intercepts.

(c) Find all vertical/horizontal/slant asymptotes.

(d) Find where f is increasing and where it is decreasing. Classify all critical points.

(e) Find where f is concave up/down. Identify all inflection points.

(f) Use parts (a)−(e) to sketch $f(x)$.

7 Optimization Problems

7.1 VIDEO - Intro and Practice

Objective(s):

- Analyze real world problems and transform statements into mathematical equations.
- Apply our maxima/minima knowledge to help solve optimization problems.

Remark 7.1. Related rates problems are to ______________________________________ as optimization problems are to _____________ and ______________ problems.

Example 7.2. Find the maximum area of a rectangle inscribed in an equilateral triangle of side length 6 and one side of the rectangle lies along the base of the triangle.

Theorem 7.3 (Steps in Solving Optimization Problems).

1. ____________________ the problem.
   - Read the problem through in its entirety
   - Determine what is given and what is unknown
2. Draw a _______________
   - This is useful in most problems
3. Introduce _______________
   - Assign symbols to what needs maximized or minimized.
   - Select symbols for other quantities and label the diagram when appropriate.
4. Find an _______________ that relates the quantities with what needs to be maximized/minimized
5. Use ______________________________________ to reduce down to one variable (when applicable)
6. Use the methods in 3.1/3.3 to find an absolute maximum/minimum. In particular, if the domain is closed then the Closed Interval Method in Section 3.1 can be used.

Example 7.4. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

7.2 VIDEO - A Little Bit Harder Now

Objective(s):

- Analyze real world problems and transform statements into mathematical equations.
- Apply our maxima/minima knowledge to solve optimization problems.

Example 7.5. A fish tank with a square base is to be made of glass sides, plastic on the base, and an open top. The fish tank needs to hold 5 cubic feet of water. Glass costs \$3 per square foot and plastic cost \$2 per square foot. What is the cheapest the tank can cost?

8 Newton's Method

8.1 VIDEO - Introduction and Explanation

Objective(s):

- Develop Newton's Method.
- Apply Newton's Method to help find roots of equations.

In this section we explore a powerful algorithm that relies on tangent lines. To help us understand its value consider the following example

Example 8.1. Solve $x^5 + x - 1 = 0$

Example 8.2. Use `https://www.desmos.com/calculator/ymdbwwmyga` to approximate the solution for $x^5 + x - 1 = 0$.

Remark 8.3. As we saw above Newton's method is an powerful technique used in finding roots (zeros) of equations

Remark 8.4. In general the convergence is quadratic: as the method converges on the root, the difference between the root and the approximation is squared (the number of accurate digits roughly doubles) at each step. However, there are some difficulties with the method which we will discuss in the next video.

Remark 8.5 (Idea behind Newton's Method)**.**

1. Take an initial guess for the root of $f(x)$. Call it x_1.
2. Find the tangent line for $f(x)$ through x_1.
3. Find where the tangent line has a root and make that your next guess. Call it x_2.
4. Repeat as desired.

Pictures:

Algebraically this becomes

Theorem 8.6 (Newton's Method)**.** If x_1 is the initial guess of some root of $f(x)$ then

Example 8.7. Approximate the root of the function $f(x) = x^6 + 3x + 2$. Using $x_1 = 0$ as the starting value for Newton's method, find the next two approximations, x_2 and x_3. You do not have to simplify.

8.2 VIDEO - Issues with Newton's Method

Objective(s):

- Recognize a few ways that Newton's Method can fail.

Example 8.8. Consider the following graphs and starting points. Classify what goes wrong in each case

(a) `https://www.desmos.com/calculator/vdjn21linv`

(b) `https://www.desmos.com/calculator/malzbmcx3m`

(c) `https://www.desmos.com/calculator/pkmmrbnj36`

Remark 8.9. Newton's method may fail to converge to an answer or may find the wrong root. See the pictures below for how this can happen. When this occurs it can usually be fixed by selecting an alternative initial guess (x_1).

9 Antiderivatives

9.1 VIDEO - Definitions and Intro

Objective(s):

- Compute general antiderivatives for many types of functions.

Definition(s) 9.1. A function F is called an ____________________________ of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example 9.2. Find an antiderivative of $f(x) = 2x + 3$.

Example 9.3. Find another antiderivative of $f(x) = 2x + 3$.

Theorem 9.4. If ______________________________________ of f on an interval I, then the most general antiderivative of f on I is

where C is an arbitrary constant.

Example 9.5. Find the most general antiderivative of the functions below.

(a) $f(x) = \cos(x) + \dfrac{1}{x^2}$

(b) $f(x) = \sqrt{x}(6 + 7x)$

9.2 VIDEO - Initial Value Problems and Applications

Objective(s):

- Solve initial value problems for particular antiderivative functions.
- Use antiderivatives to calculate velocity or position from acceleration.

Example 9.6. Suppose a ball is thrown up in the air. Its velocity is given by $v(t) = 7 - 10t$ meters per second, t seconds after the ball is released. If the ball is initially 1 meter above the ground find the ball's position function, $s(t)$.

Definition(s) 9.7. A ____ is an equation involving the derivatives of an unknown function.

Definition(s) 9.8. An ____ is a differential equation for $y = f(x)$ along with an ________________________________, such as $f(c) = a$ for some constants c and a. The solution to the initial value problem is a solution to the differential equation that also satisfies the initial condition.

Example 9.9. Solve the initial value problem: $f'(x) = 1 + 3\sqrt{x}, \quad f(4) = 25$

Example 9.10. A particle is moving with velocity $v(t) = \sin t - \cos t$, and has initial position $s(0) = 0$. Find the position function of the particle.