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Chapter 5

Applications of Integration
5.2 Volumes

Section Objective(s):

- Use integrals to find the volume of a 3D solid
- Identify whether to use the disk, washer, or another method to find volume

Example 5.2.1. Let \( R \) be the region bounded by \( y = x^2 \), \( x = 1 \), and the \( x \)-axis. Find the area of \( R \) by integrating

(a) with respect to \( x \).

(b) with respect to \( y \).
Example 5.2.2. Let $R$ be the region bounded by $y = x^2$, $x = 1$, and the $x$-axis. Find the volume of the solid generated when $R$ is rotated around the $x$-axis.

Definition(s) 5.2.3. Let $S$ be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of $S$ in the plane $P_x$, through $x$ and perpendicular to the $x$-axis, is $A(x)$, where $A$ is a continuous function, then the ____________ of $S$ is
Definition(s) 5.2.4. The solid in the previous example is called a ____________________________
because it is obtained by revolving a region about a line.

Theorem 5.2.5. In general we can calculate the volume of a solid of revolution by using the basic defining
formula

\[ V = \quad \text{or} \quad V = \]

and we find the cross-sectional area \( A(x) \) or \( A(y) \) in one of the following ways:

- If the cross-section is a disk, we find the radius of the disk(in terms of \( x \) or \( y \)) and use

  \[ A = \]

- If the cross-section is a washer, we find the inner radius \( r_{\text{in}} \) and outer radius \( r_{\text{out}} \) from a sketch and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

  \[ A = \]

Example 5.2.6. Let \( R \) be the region bounded by \( y = x^2, x = 1 \), and the \( x \)-axis. Find the volume of the solid generated when \( R \) is rotated around the \( y \)-axis.
Example 5.2.7. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}, x = 0, y = 2,$ and $x = 1$ about the line $y = 3$. Draw a picture of the volume you are finding.

Example 5.2.8. Consider the region of the $xy$-plane bounded by $y = 0, x = 0$ and $y = 1 - x$. Find the volume of the solid generated by revolving this region about the line $x = 2$. 
Example 5.2.9. Consider the solid with triangular base formed by $y = \frac{x}{2}$, $y = 0$ and $x = 4$, for which parallel cross-sections perpendicular to the base and $x$-axis are squares. Find the volume of such a solid (shown below).

![Diagram of Example 5.2.9](image)

Example 5.2.10. Find the volume of the solid whose base $B$ is the region bounded by the parabola $y = x^2$ and $y = 1$ and whose cross sections perpendicular to the $y$-axis are equilateral triangles.

![Diagram of Example 5.2.10](image)
5.4 Work

Section Objective(s):

- Use integrals to find work
- Determine the equation for force in a given situation

Definition(s) 5.4.1. We define the ______ done in moving an object from ______ as:

Example 5.4.2. When a particle is located a distance $x$ feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?
Example 5.4.3. A 10 ft cable weighing 50 lbs and a bucket of water weighing 60 lbs is hanging down in a well. How much work must be done to lift bucket and cable to the top of the well?

Definition(s) 5.4.4. Hooke's Law states that the force required to maintain a spring stretched $x$ units beyond its natural length is proportional to $x$. That is: $f(x) = kx$ where $k$ is a positive constant.

Example 5.4.5. A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 14 cm to 16 cm?
Example 5.4.6. A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with oil weighing 30 pounds per cubic foot to a height of 8 m. Find the work required to empty the tank by pumping all of the oil to the top of the tank.

Theorem 5.4.7. If $\sigma$ is the density of the liquid, $d(y)$ is the distance to the where the liquid is being pumped, $A(y)$ is the area of a cross selection of liquid perpendicular to the $y$-axis and the liquid exists between $y = a$ and $y = b$ then

$$\text{Work} =$$
Example 5.4.8. A tank has the shape of an inverted right rectangular cone with height 8 m and base rectangle 4m × 4m shown below. It is filled with oil weighing 20 N/m³. Find the work required to pump the oil to a spot 3 meters above the tank.
Chapter 6

Inverse Functions:

Exponential, Logarithmic, and Inverse Trigonometric Functions
Forest for the Trees

Months from now you will look back and long for the simpler times of Chapter 6. In Chapter 6 we add a few more functions to our calculus tool kit including how to differentiate and integrate functions related to:

- Exponentials, Logarithms, Inverse Trigonometrics, and Hyperbolics

In addition, we will learn how to differentiate in style by using a technique called logarithmic differentiation. We will also study how to defeat more interesting limits using L’Hospital’s Rule. By the end of Chapter 6 you will be able to solve problems such as:

Example: Find the derivatives of:

(a) \( f(x) = \ln x + \log_5 x \)  
(b) \( g(x) = e^x + 2^x \)  
(c) \( h(x) = (2x)^{3x} \)  
(d) \( h(x) = \sin^{-1}(x) + \tanh(x) \)

Example: Bill Bourbon was murdered in Savannah sometime last night. The temperature outside has been a constant 65°F. At 9AM Bill’s body temperature was 82°F and at 10AM his temperature was 80°F. Nathaniel Nutmeg was seen at the bar from 9PM-2AM last night. Could he have committed the murder? (Assume initial body temperature was 98.6°F.)

Example: Evaluate the following limits:

(a) \( \lim_{x \to 0^+} \left( \frac{1}{\sin x} - \frac{1}{x} \right) \)  
(b) \( \lim_{x \to 0^+} (2x)^{3x} \)  
(c) \( \lim_{x \to 0^+} \left( \frac{3}{x} \right)^{5x} \)  
(d) \( \lim_{x \to \infty} (1 + \frac{2}{x})^x \)
6.1 Inverse Functions

Section Objective(s):

- Find the inverse of a function
- Evaluate the derivative of the inverse of a given function

Definition(s) 6.1.1. A function \( f \) is called a \( \underline{\text{one-to-one function}} \) if it never takes on the same value twice; that is,

\[ f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2 \]

Remark 6.1.2. \( \underline{\text{Horizontal Line Test}} \): A function is one-to-one if and only if no horizontal line intersects its graph \( \underline{\text{___________________________}} \)

Definition(s) 6.1.3. Let \( f \) be a one-to-one function with domain \( A \) and range \( B \). Then its \( \underline{\text{inverse function}} \) has domain \( B \) and range \( A \) and is defined by

\[ f^{-1}(y) = x \iff f(x) = y \]

for any \( y \) in \( B \).

Remark 6.1.4. Note that:

- domain of \( f^{-1} = \text{range of } f \)
- range of \( f^{-1} = \text{domain of } f \)
CHAPTER 6. INVERSE FUNCTIONS:

Definition(s) 6.1.5. Cancellation equations:

\[ f^{-1}(f(x)) = x \text{ for every } x \text{ in } A \]

\[ f(f^{-1}(x)) = x \text{ for every } x \text{ in } B \]

Theorem 6.1.6. How to Find the Inverse Function of a One-to-One Function \( f \):

**Step 1** Write \( y = f(x) \).

**Step 2** Solve this equation for \( x \) in terms of \( y \) (if possible).

**Step 3** To express \( f^{-1} \) as a function of \( x \), \[ \text{expression} \]. The resulting equation is \( y = \text{expression} \).

Example 6.1.7. Find the inverse function of \( f(x) = x^3 + 2 \).

Remark 6.1.8. The graph of \( f^{-1} \) is obtained by reflecting the graph of \( f \) about the line \( y = x \).
Theorem 6.1.9. If $f$ is a one-to-one continuous function defined on an interval, then its inverse function $f^{-1}$ is continuous.

Theorem 6.1.10. If $f$ is a one-to-one differentiable function with inverse function $f^{-1}$ and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at $a$ and $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$.

Example 6.1.11. If $f(x) = x^3 + 2$, find $(f^{-1})'(3)$. 


Example 6.1.12. Consider the function $f(x) = \frac{4x + 1}{2x + 3}$

(a) Find the inverse of $f$

(b) Find $(f^{-1})'(1/3)$
6.2 The Natural Logarithmic Function

Section Objective(s):
- Identify properties of the natural logarithmic function.
- Explore the process of logarithmic differentiation

Definition(s) 6.2.1. The natural logarithmic function is the function defined by

Theorem 6.2.2. Laws of Logarithms If $x$ and $y$ are positive numbers and $r$ is a rational number, then

(a) $\ln(xy) =$

(b) $\ln\left(\frac{x}{y}\right) =$

(c) $\ln(x^r) =$

Example 6.2.3. Express $\ln(3) + 4 \ln(4x)$ as a single logarithm.
Remark 6.2.4.

\[
\frac{d}{dx}(\ln x) = \]

Definition(s) 6.2.5. __ is the number such that ______________.

Remark 6.2.6. In general, if we combine Remark 6.2.2 with the Chain Rule, we get

Example 6.2.7. Find the derivative \( \frac{d}{dx} \ln(\sin(3x^2)) \).
### Remark 6.2.8.

\[
\frac{d}{dx} (\ln |x|) = \int \frac{1}{x} \, dx = \int \tan x \, dx =
\]

### Theorem 6.2.9. Steps in Logarithmic Differentiation

1. Take \( \ln \) of both sides of an equation \( y = f(x) \) and use the \( \ln \) to simplify.

2. \( \tan \) implicitly with respect to \( x \).

3. Solve the resulting equation for \( y \).
Example 6.2.10. Compute the derivative for the following function: \[ y = \frac{(\sin^2 x)(x + 3)^4}{(5x - 8)^{10}} \]
Example 6.2.11. Find the derivative of \( y = t^2 + 3 \ln(5 \ln(t)) \)

Example 6.2.12. Evaluate the integral \( \int_{9}^{10} \frac{1}{t \ln t} \, dt \)


6.3 The Natural Exponential Function

Section Objective(s):

- Identify properties of the natural exponential function.
- Practice differentiation/integration of the natural exponential function.

Definition(s) 6.3.1. The inverse of the \( \text{exp}(x) \) is denoted by \( e^x \). We define

\[ e^x = y \iff \ln y = x \]

Cancellation equations:

Example 6.3.2. Solve the equation \( 2e^{2-x} = 30 \).
Theorem 6.3.3. Properties of the Natural Exponential Function  The exponential function
\[ f(x) = e^x \]
is an increasing continuous function with domain \( \mathbb{R} \) and range \((0, \infty)\). Thus ________
for all \( x \). Also

So the \( x \)-axis is a horizontal ________ of \( f(x) = e^x \).

Definition(s) 6.3.4. Laws of Exponents  If \( x \) and \( y \) are real numbers and \( r \) is rational, then

(a) \( e^{x+y} = \)

(b) \( e^{x-y} = \)

(c) \( (e^x)^r = \)
Remark 6.3.5. The natural exponential function has the remarkable property that it is its own derivative.

In general, if we combine this with the Chain Rule, we get

$$\frac{d}{dx}(e^u) =$$

Example 6.3.6. Evaluate the derivative $\frac{d}{dx}e^{\sin(3x)}$.

Example 6.3.7. $\int e^x \, dx$
Example 6.3.8. Find the equation of the tangent line to the curve \( y = \frac{4e^{-x}}{x^2} \) at \((-2, e^2)\).
6.4 General Logarithmic and Exponential Functions

Section Objective(s):
- Explore the properties and calculus of general exponential functions.
- Explore the properties of the general logarithmic function.

Definition(s) 6.4.1. We define

\[ a^x = \]

Theorem 6.4.2. The general laws of exponents follow from this together with the laws for \( e^x \). If \( x \) and \( y \) are real numbers and \( a, b > 0 \), then

(a) \( a^{x+y} = \)

(b) \( a^{x-y} = \)

(c) \( (a^r)^x = \)

(d) \( (ab)^x = \)

Theorem 6.4.3.

(a) \( \frac{d}{dx} (a^x) = \)

(b) \( \int a^x \, dx = \) \quad when \( a \neq 1 \)
Example 6.4.4. Evaluate $\int 4^{-x} dx$.

Theorem 6.4.5. The Power Rule  If $n$ is any real number and $f(x) = x^n$, then

Definition(s) 6.4.6. If $a > 0$ and $a \neq 1$, then $f(x) = ____$ is a one-to-one function. Its inverse function is called the ________________ function with ________ and is denoted by _______. Thus

Remark 6.4.7. Change of Base Formula For any positive number $a$ with $a \neq 1$, we have

Example 6.4.8. Evaluate the following: $\log_8 256$
CHAPTER 6. INVERSE FUNCTIONS:

Theorem 6.4.9. \( \frac{d}{dx}(\log_a x) = \)

Example 6.4.10. Use logarithmic differentiation to find \( \frac{dy}{dx} \) for \( y = (1 + x)^{1/x} \)
Example 6.4.11. Find the derivative of

(a) \( y = 4^x + x^4 \)

(b) \( f(x) = 3^{\sin(2x)} \)

(c) \( g(x) = (2x)^{3x} \quad \text{for} \; x > 0 \)
Example 6.4.12. Evaluate the integrals

(a) \[ \int x^5 \cdot 2 \, dx \]

(b) \[ \int 2^{\sin \theta} \cos \theta \, d\theta \]
6.5 Exponential Growth and Decay

Section Objective(s):

- Explore different applications of exponential functions.
- Recognize exponentials are the solutions to certain differential equations.

**Theorem 6.5.1.** The only solutions of the differential equation \( \frac{dy}{dt} = ky \) are the exponential functions

\[ y(t) = \]

**Definition(s) 6.5.2.** In the context of population growth, where \( P(t) \) is the size of a population at time \( t \), we can write

or

This \( k \) is the growth rate divided by the population size; it is called the relative growth rate. If the population at time 0 is \( P_0 \), then the expression for the population is

\[ P(t) = \]

**Example 6.5.3.** The number of cases of a disease is reduced by 20% each year. If there are 10,000 cases today, how long will it take to reduce the number to 1000?
Definition(s) 6.5.4. Radioactive substances decay by spontaneously emitting radiation. If \( m(t) \) is the mass remaining from an initial mass \( m_0 \) of the substance after time \( t \)

where \( k \) is a negative constant. The mass decays exponentially:

\[
m(t) = m_0 e^{-kt}
\]

Example 6.5.5. Element \( X \) is radioactive. 7 days ago my sample of element \( X \) weighed 100 grams but today it only weighs 90 grams. How many days until it weighs 45 grams?

Definition(s) 6.5.6. Physicists express the rate of decay in terms of half-life, the time required for half of any given quantity to decay. This can be found by

\[
\text{half-life} = \frac{-\ln 2}{k}
\]
Definition(s) 6.5.7. **Newton’s Law of Cooling** states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. If we let \( T(t) \) be the temperature of the object at time \( t \) and \( T_s \) be the temperature of the surroundings, then

\[
\frac{dT}{dt} = -k(T - T_s)
\]

where \( k \) is a constant. This can be solved to find

\[
T(t) = T_s + (T_0 - T_s)e^{-kt}
\]

**Example 6.5.8.** Bill Bourbon was murdered in Savannah sometime last night. The temperature outside has been a constant 65° F. At 9AM Bill’s body temperature was 82° F and at 10AM his temperature was 80° F. Nathaniel Nutmeg was seen at the bar from 9PM-2AM last night. Could he have committed the murder? *(Assume initial body temperature was 98.6° F.)*
9.3 Separable Equations

Section Objective(s):

- Apply techniques to solve separable differential equations.
- Recall how to solve initial value problems.

Definition(s) 9.3.1. A ____________________________ is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be written in the form:

Equivalently, if ______________ then we could write it as:

where $h(y) = 1/g(y)$.

Technique for solving separable differential equations

1. Rewrite the equation in the differential form:
2. Integrate both sides:
3. Solve for $y$ in terms of $x$ (if possible)

Example 9.3.2. Find the general solution of the differential equation $2yy' - x = 0$
Example 9.3.3. Solve the initial value problems:

(a) \( y \frac{dy}{dx} - e^x = 0, \quad y(0) = 4 \).

(b) \( 2x \frac{dy}{dx} - \ln x^2 = 0, \quad y(1) = 2 \).

(c) \( \frac{dT}{dt} = -k(T - T_s), \quad T(0) = T_0 \quad \text{(to verify the formula for Newton’s Law of Cooling)} \)
6.6 Inverse Trigonometric Functions

Section Objective(s):

- Define the inverse trigonometric functions
- Understand the calculus of the inverse trigonometric functions

**Definition(s) 6.6.1.** The \( \arcsin x \), \( \sin^{-1} x \), has domain \([-1, 1]\) and range \([-\pi/2, \pi/2]\)

\[ \arcsin x = y \iff \sin y = x \quad \text{and} \quad -\pi/2 \leq y \leq \pi/2 \]

The cancellation equations for inverse functions become, in this case,

- for \( -\pi/2 \leq x \leq \pi/2 \):
  \[ \sin(\arcsin x) = x \]
- for \( -1 \leq x \leq 1 \):
  \[ \arcsin(\sin x) = x \]

and its derivative is given by

\[ \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1 \]

**Example 6.6.2.** Evaluate \( \frac{d}{dx} \arcsin(\ln(3x^2)) \)
Definition(s) 6.6.3. The inverse \textbf{function} is handled similarly.

\[
\cos^{-1} x = y \iff \cos y = x \text{ and } 0 \leq y \leq \pi
\]

The cancellation equations for inverse functions become, in this case,

for \(0 \leq x \leq \pi\)

for \(-1 \leq x \leq 1\)

The inverse cosine function, \(\cos^{-1}\), has domain \([-1, 1]\) and range \([0, \pi]\), and its derivative is given by

\[-1 < x < 1\]

Definition(s) 6.6.4. The inverse \textbf{function} is defined

\[
\tan^{-1} x = y \iff \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}
\]

The lines \(\text{___________}\) and \(\text{___________}\) are horizontal asymptotes of the graph of \(\tan^{-1}\).

Its derivative is given by

\[
\frac{d}{dx}(\tan^{-1} x) = \quad
\]
CHAPTER 6. INVERSE FUNCTIONS:

Remark 6.6.5. Pictures of graphs and derivatives

\[ \sin^{-1} x \]  \[ \cos^{-1} x \]  \[ \tan^{-1} x \]

\[ (\sin^{-1} x)' \]  \[ (\cos^{-1} x)' \]  \[ (\tan^{-1} x)' \]

Example 6.6.6. Evaluate \( \frac{d}{dx} \cos^{-1}(5^x) \)

Example 6.6.7. Evaluate \( \int \frac{3}{(2x)^2 + 1} \, dx \)
Theorem 6.6.8. The remaining inverse trigonometric functions are not used as frequently and are summarized here.

\[ y = \csc^{-1}(x) \quad (|x| \geq 1) \quad \iff \]
\[ y = \sec^{-1}(x) \quad (|x| \geq 1) \quad \iff \]
\[ y = \cot^{-1}(x) \quad (x \in \mathbb{R}) \quad \iff \]

Theorem 6.6.9. Table of Derivatives of Inverse Trigonometric Functions

\[ \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \]
\[ \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x \sqrt{1-x^2}} \]
\[ \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \]
\[ \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x \sqrt{x^2-1}} \]
\[ \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \]
\[ \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2} \]

Theorem 6.6.10.

\[ \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C \]
\[ \int \frac{1}{x^2+1} dx = \tan^{-1} x + C \]

Example 6.6.11. \[ \int \frac{1}{x^2 + a^2} dx \]
Example 6.6.12. Evaluate: \[
\int \frac{dx}{x^2 + 2x + 5}.
\] (Hint: complete the square)

Example 6.6.13. Evaluate: \[
\int \frac{x + 1}{1 + x^2} \, dx.
\]
6.7 Hyperbolic Functions

**Section Objective(s):**
- Introduce and define the hyperbolic functions
- Examine the properties of the hyperbolic functions

**Definition(s) 6.7.1. Definition of the Hyperbolic Functions**

\[
\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}
\]

\[
\text{csch}(x) = \frac{1}{\sinh(x)} \quad \text{sech}(x) = \frac{1}{\cosh(x)}
\]

\[
\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \quad \coth(x) = \frac{\cosh(x)}{\sinh(x)}
\]

**Remark 6.7.2.** Graphs of hyperbolic sine and cosine

\[
\sinh x \quad \cosh x
\]
Theorem 6.7.3. Hyperbolic Identities

\[
\sinh(-x) = -\sinh x \\
\cosh(-x) = \cosh x \\
\cosh^2 x - \sinh^2 x = 1 - \tanh^2 x =
\]

Remark 6.7.4. the relations between \(\sinh t\) and \(\sin t\), \(\cosh t\) and \(\cos t\).

Theorem 6.7.5. Table of Derivatives of Inverse Trigonometric Functions

\[
\frac{d}{dx}(\sinh x) = \cosh x \\
\frac{d}{dx}(\cosh x) = \sinh x \\
\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x} \\
\frac{d}{dx}(\operatorname{csch} x) = -\coth x \\
\frac{d}{dx}(\operatorname{sech} x) = \frac{\tanh x}{\cosh^2 x} \\
\frac{d}{dx}(\operatorname{coth} x) = -\frac{1}{\operatorname{csch}^2 x}
\]
Example 6.7.6. Evaluate the following derivatives:

(a) \[ \frac{d}{dx} \tanh(\sqrt{1 + x^2}) \]

(b) \[ \frac{d}{d\theta} \sinh(\ln(\cosh \theta)) \]

Example 6.7.7. Evaluate the limit: \[ \lim_{x \to \infty} \frac{\sinh(2x)}{\cosh(3x)}. \]
Example 6.7.8. Evaluate the following integrals:

(a) \( \int \sinh(3x + 1) \, dx \)

(b) \( \int \tanh(4x) \, dx \)

(c) \( \int_0^1 t^3(\cosh^2(5t) - \sinh^2(5t)) \, dt \)
6.8 Indeterminate Forms and L’Hospital’s Rule

Section Objective(s):
- Explore indeterminate forms of limits
- Understand and apply L’Hospital’s Rule

Definition(s) 6.8.1. If we have a limit of the form

\[ \lim_{x \to a} f(x) / g(x) \]

where both \( f(x) \to 0 \) and \( g(x) \to 0 \) as \( x \to a \), then the limit may or may not exist and is called an indeterminate form \( \frac{0}{0} \).

If we have a limit of the form

\[ \lim_{x \to a} f(x) / g(x) \]

where both \( f(x) \to \infty \) (or \( -\infty \)) and \( g(x) \to \infty \) (or \( -\infty \)) as \( x \to a \), then the limit may or may not exist and is called an indeterminate form of \( \frac{\infty}{\infty} \).

Theorem 6.8.2. L’Hospital’s Rule Suppose \( f \) and \( g \) are differentiable and and \( g'(x) \neq 0 \) on an open interval \( I \) that contains \( a \) (except possibly at \( a \)). Suppose that

and

or that

\[ \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \text{finite} \]

(In other words, we have an indeterminate form of type \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \).) Then

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \]

if the limit on the right side exists (or is \( \infty \) or \( -\infty \)).
Example 6.8.3. Use L’Hospital’s Rule to calculate \( \lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} \).

Remark 6.8.4. There are additional indeterminate forms such as:

We will demonstrate how to utilize L’Hospital’s Rule to solve these.

Calculus 1 Theorem: if \( f(x) \) is continuous then:

if the limit exists.
Example 6.8.5. Use L’Hospital’s Rule to calculate the following limits:

(a) \( \lim_{x \to \infty} \frac{1}{x} \ln x \)

(b) \( \lim_{x \to 0^+} \left( \frac{1}{\sin x} - \frac{1}{x} \right) \)

(c) \( \lim_{x \to 0^+} (2x)^{3x} \)
Example 6.8.6. Use L'Hospital's Rule to calculate the following limits:

(a) \[ \lim_{x \to 0^+} \left( \frac{3}{x} \right)^{5x} \]

(b) \[ \lim_{x \to \infty} \left( \frac{3 - x}{x} \right)^{5x} \]

(c) \[ \lim_{x \to \infty} \left( 1 + \frac{2}{x} \right)^x \]
Chapter 7

Techniques of Integration
CHAPTER 7. TECHNIQUES OF INTEGRATION

Forest for the Trees

Chapter 7 is all about integration. We will learn quite a few new tricks such as Integration by Parts, Trigonometric Substitution, Partial Fractions, and Improper Integrals. In fact we will learn so many tricks/techniques that it will be hard to keep them all straight. Because of this we will end the Chapter with 7.5, Strategy for Integration, which helps us strategise what we should when we encounter a new integral.

By the end of Chapter 7 you should be able to solve problems such as:

Example : Evaluate $\int \frac{\sqrt{4 - x^2}}{x^2} dx$

Example : Evaluate $\int e^x \cosh 2x \, dx$

Example : Evaluate $\int_{-1}^{1} \frac{1}{x^2} \, dx$
7.1 Integration by Parts

Section Objective(s):

- Learn the method of integration by parts
- Apply integration by parts to a variety of problems

Theorem 7.1.1. Formula for integration by parts

\[ \int f(x)g'(x)dx = \]

Let \( u = f(x) \) and \( v = g(x) \). By the Substitution Rule, the formula becomes

Remark 7.1.2. We can evaluate definite integrals by parts:

\[ \int_a^b f(x)g'(x)dx = \]
Example 7.1.3. Evaluate $\int x \cos x \, dx$

Example 7.1.4. Evaluate $\int x^2 e^{3x} \, dx$
Example 7.1.5. Evaluate $\int_1^2 \ln x \, dx$

Example 7.1.6. Evaluate $\int \tan^{-1}(5x) \, dx$
Example 7.1.7. Evaluate $\int e^x \cosh 2x \, dx$
7.2 Trigonometric Integrals

Section Objective(s):
- Identify strategies for integrating certain combinations of trigonometric functions
- Integrate some integrals!

Theorem 7.2.1. Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx =$$

Then substitute $u =$

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx =$$

Then substitute $u =$

Example 7.2.2. Evaluate $\int \sin^4 x \cos^3 x \, dx$
Theorem 7.2.1 Continued:

(c) If the powers of both sine and cosine are even, use the half-angle identities

\[ \sin^2 x = \quad \cos^2 x = \]

It is sometimes helpful to use the identity

\[ \sin x \cos x = \]

Example 7.2.3. Evaluate \( \int \sin^2 x \cos^2 x \, dx \)
Theorem 7.2.4. Strategy for Evaluating \( \int \tan^m x \sec^n x \, dx \)

(a) If the power of secant is even \((n = 2k, k \geq 2)\), save a factor of \(\sec^2 x\) and use \(\sec^2 x = 1 + \tan^2 x\) to express the remaining factors in terms of \(\tan x\):

\[
\int \tan^m x \sec^{2k} x \, dx = \]

Then substitute \(u = \)

(b) If the power of tangent is odd \((m = 2k + 1)\), save a factor of \(\sec x \tan x\) and use \(\tan^2 x = \sec^2 x - 1\) to express the remaining factors in terms of \(\sec x\):

\[
\int \tan^{2k+1} x \sec^n x \, dx = \]

Then substitute \(u = \)

Example 7.2.5. Evaluate \( \int \tan^4 x \sec^6 x \, dx \)
Theorem 7.2.6. We will sometimes need to be able to integrate $\tan x$ or $\sec x$:

- $\int \tan x \, dx = \ln|\sec x| + C$
- $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

Proof:
Example 7.2.7. Evaluate $\int \tan^3 x \, dx$

Example 7.2.8. Evaluate $\int x \sin^3 x \, dx$
Theorem 7.2.9. To evaluate the integrals

(i) \( \int \sin(mx) \cos(nx) \, dx \),

(ii) \( \int \sin(mx) \sin(nx) \, dx \),

(iii) \( \int \cos(mx) \cos(nx) \, dx \),

use the corresponding identities:

(a) \( \sin A \cos B = \)

(b) \( \sin A \sin B = \)

(c) \( \cos A \cos B = \)

Example 7.2.10. Evaluate \( \int \sin(2x) \cos(3x) \, dx \)
7.3 Trigonometric Substitution

Section Objective(s):
- Understand the trigonometric substitution technique for integrating certain functions
- Identify integrals problems that can be solved with this technique.

Definition(s) 7.3.1. In general, we can make a substitution of the form \( x = g(t) \) by using the Substitution Rule in reverse. To make our calculations simpler, we assume that \( g \) has an inverse function; that is, \( g \) is one-to-one. In this case, if we replace \( u \) by \( x \) and \( x \) by \( t \) in the Substitution Rule, we obtain

\[
\int f(x) \, dx = \int f(g(t)) \, g'(t) \, dt
\]

This kind of substitution is called ________________

Theorem 7.3.2. Table of Trigonometric Substitutions

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Substitution</th>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{a^2 - x^2} )</td>
<td>( x = a \sin \theta )</td>
<td>(- \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})</td>
</tr>
<tr>
<td>( \sqrt{a^2 + x^2} )</td>
<td>( x = a \tan \theta )</td>
<td>(- \frac{\pi}{2} &lt; \theta &lt; \frac{\pi}{2})</td>
</tr>
<tr>
<td>( \sqrt{x^2 - a^2} )</td>
<td>( x = a \sec \theta )</td>
<td>(- 0 \leq \theta &lt; \frac{\pi}{2}) or ( \pi \leq \theta &lt; \frac{3\pi}{2})</td>
</tr>
</tbody>
</table>
Example 7.3.3. Evaluate \[ \int \frac{dx}{\sqrt{4 + x^2}} \]
Example 7.3.4. Evaluate $\int \frac{dx}{\sqrt{9 - x^2}}$
Example 7.3.5. Evaluate \( \int \frac{\sqrt{4-x^2}}{x^2} \, dx \)
Example 7.3.6. Evaluate $\int \frac{\sqrt{x^2 - 25}}{x} \, dx$
7.4 Integration of Rational Functions by Partial Fractions

Section Objective(s):
- Understand how to break apart fractions in a new way.
- Recall how to solve systems of equations.
- Learn the partial fraction method of integrating rational functions

Definition(s) 7.4.1. Partial fractions can be used to help integrate many functions. It essentially “finding a common” in reverse.

Example 7.4.2. Use the fact that \( \frac{2x + 1}{x(x + 1)} = \frac{1}{x} + \frac{1}{1 + x} \) to evaluate \( \int \frac{2x + 1}{x^2 + x} \, dx \)

Partial Fractions Technique: for the rational function \( \frac{f(x)}{g(x)} \) where \( \deg(f) < \deg(g) \)

1. Factor the into and irreducible terms.

2. Express the rational function as a sum of partial fractions of the form

or

where \( i \) takes on less than or equal to the power of the factor.

Example 7.4.3. Evaluate \[ \int \frac{5x - 3}{x^2 - 2x - 3} \, dx \]

Example 7.4.4. Evaluate \[ \int \frac{x + 4}{(x + 1)^2} \, dx \]
Example 7.4.5. Evaluate \[ \int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} \, dx \]

\[ \begin{align*}
\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} &= \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2} \\
&= \frac{(Ax + B)(x - 1)^2 + C(x^2 + 1)(x - 1) + D(x^2 + 1)}{(x^2 + 1)(x - 1)^2} \\
&= \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}
\end{align*} \]

Remark 7.4.6. To apply partial fractions for rational functions \( \frac{f(x)}{g(x)} \) where \( \deg(f(x)) \geq \deg(g(x)) \), include the additional step

0. \( \frac{f(x)}{g(x)} \) the rational function using polynomial \( \frac{Ax + B}{x^2 + 1} \) to express it as

the sum of a \( \frac{C}{x - 1} \) and a proper rational function.
Example 7.4.7. Evaluate \[ \int \frac{x^5 + 5x^4 + 7x^3 + x + 2}{x^4 + x^2} \, dx \]
7.8 Improper Integrals

Example 7.8.1. Evaluate $\int_{-1}^{1} \frac{1}{x^2} \, dx$

Section Objective(s):
- Understand the need for improper integrals
- Classify different types of improper integrals
- Identify when improper integrals are needed
- Evaluate some improper integrals
Definition(s) 7.8.2. Improper Integral of Type 1

(a) If \( \int_{a}^{t} f(x) \, dx \) exists for every number \( t \geq a \), then

\[
\int_{a}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{a}^{t} f(x) \, dx
\]

provided this limit exists (as a finite number).

(b) If \( \int_{t}^{b} f(x) \, dx \) exists for every number \( t \leq b \), then

\[
\int_{-\infty}^{b} f(x) \, dx = \lim_{t \to -\infty} \int_{t}^{b} f(x) \, dx
\]

provided this limit exists (as a finite number).

The improper integrals \( \int_{a}^{\infty} f(x) \, dx \) and \( \int_{-\infty}^{b} f(x) \, dx \) are called ________________ if the corresponding limit exists and ________________ if the limit does not exist.

(c) If both \( \int_{a}^{\infty} f(x) \, dx \) and \( \int_{-\infty}^{a} f(x) \, dx \) are convergent, then we define

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{\infty} f(x) \, dx
\]

In part (c) any real number \( a \) can be used.
Example 7.8.3. Determine if the following integral converges or diverges: \[ \int_{1}^{\infty} \frac{\ln x}{x^2} \, dx \]

Theorem 7.8.4. \[ \int_{1}^{\infty} \frac{1}{x^p} \, dx \] is convergent if \( p > 1 \) and divergent if \( p \leq 1 \).

Proof:
Definition(s) 7.8.5. Improper Integral of Type 2

(a) If \( f \) is continuous on \([a, b)\) and is discontinuous at \( b \), then

\[
\int_{a}^{b} f(x) \, dx = \lim_{t \to b^-} \int_{t}^{a} f(x) \, dx
\]

if this limit exists (as a finite number).

(b) If \( f \) is continuous on \((a, b]\) and is discontinuous at \( a \), then

\[
\int_{a}^{b} f(x) \, dx = \lim_{t \to a^+} \int_{a}^{b} f(x) \, dx
\]

if this limit exists (as a finite number).

The improper integral \( \int_{a}^{b} f(x) \, dx \) is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If \( f \) has a discontinuity at \( c \), where \( a < c < b \), and both \( \int_{a}^{c} f(x) \, dx \) and \( \int_{c}^{b} f(x) \, dx \) are convergent, then we define

Example 7.8.6. Determine if the following integral converges or diverges: \( \int_{0}^{1} \frac{1}{\sqrt{x}} \, dx \)
CHAPTER 7. TECHNIQUES OF INTEGRATION

Theorem 7.8.7. Comparison Theorem Suppose that $f$ and $g$ are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty \text{ } dx$ is convergent, then $\int_a^\infty \text{ } dx$ is convergent.

(b) If $\int_a^\infty \text{ } dx$ is divergent, then $\int_a^\infty \text{ } dx$ is divergent.

Example 7.8.8. Determine if the following integral converges or diverges: $\int_1^\infty \frac{\sin^2 x}{x^2} \text{ } dx$
7.5 Strategy for Integration

Section Objective(s):

- Review the various formulas for integration
- Overview a general strategy for integration

Remark 7.5.1. A Four-Step Strategy for Integration

1. **if possible** Sometimes the use of algebraic manipulation or trigonometric identities will make the method of integration obvious.

2. Look for an obvious **Try to find some function \( u = g(x) \) in the integrand whose differential \( du = g'(x) \, dx \) also occurs, apart from a constant factor.

3. the integrand according to its **If steps 1 and 2 have not led to the solution, then we take a look at the form of the integrand \( f(x) \) which could include any of the following:
   
   (a)
   
   (b)
   
   (c)
   
   (d)

4. If the first three steps have not produced the answer, remember that there are basically only two methods of integration: substitution and parts.
Example 7.5.2. Evaluate \( \int_{0}^{1} \frac{5x + 1}{3x + 2} \, dx \)
Example 7.5.3. Evaluate $\int x^3 e^{x^2} \, dx$
Example 7.5.4. Evaluate \( \int \sqrt{x^2 - 2x + 2} \, dx \)
Chapter 8

Further Applications of Integration
8.1 Arc Length

Section Objective(s):
- Understand the formula for calculating arc length

Theorem 8.1.1. The Arc Length Formula
If \( f(x) \) is continuous on \([a, b]\), then the length of the curve \( y = f(x), a \leq x \leq b \), is

\[
L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx
\]

Remark 8.1.2. If a curve has the equation \( x = g(y) \) for \( c \leq y \leq d \), and \( g'(y) \) is continuous then by interchanging the roles of \( x \) and \( y \) we obtain the following formula for its length:

\[
L = \int_c^d \sqrt{1 + [g'(y)]^2} \, dy
\]

Example 8.1.3. Find the arc length for \( f(x) = \frac{x^3}{12} + \frac{1}{x} \) on the interval \( 1 \leq x \leq 4 \).
Idea of a Proof of Theorem 8.1.1

**Definition(s) 8.1.4.** If a smooth curve $C$ has the equation $y = f(x)$, for $a \leq x \leq b$, let $s(x)$ be the distance along $C$ from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$. Then $s$ is a function, called the __________, and

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} \, dt$$

**Example 8.1.5.** Find the arc length function for $f(x) = \frac{x^3}{12} + \frac{1}{x}$ starting at initial point $x = 1$. 
Example 8.1.6. Find the arc length of $f(x) = \ln(\cos x)$ on the interval $0 \leq x \leq \pi/3$.

Example 8.1.7. Setup an integral that represents the arc length function for $f(t) = \tan^{-1}(t)$ with initial point at $t = 1$. 
Chapter 11

Infinite Sequences and Series
CHAPTER 11. INFINITE SEQUENCES AND SERIES

Forest for the Trees

In Chapter 11 we switch gears to an old topic. Recall back in calculus 1 we found that the tangent line of $f$ near $a$ approximated $f$ near $a$ pretty well (we called it the linear approximation). One natural way to expand on this is ask “What quadratic (2nd degree) polynomial approximates $f$ near $a$ the best?” or how about a 3rd degree polynomial, or 4th, or 5th? Turns out with higher degree polynomials you can get better and better approximations of a function. So what about infinite polynomials? By that I mean polynomials that don’t have a highest degree that just go on and on for ever. For instance:

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$$

It turns out that sometimes you can take these infinite polynomials (better known as power series) and make them equal to the function itself! So this is our end goal. To get there we need a reminder about sequences and will be practicing with normal series for awhile before upgrading to power series!

By the end of Chapter 11 you should be able to solve problems such as:

**Example** : Evaluate Find the Maclaurin series for the function $f(x) = \sqrt{1 + x}$ and find its radius of convergence.

**Example** : Evaluate Determine if the series $\sum_{n=0}^{\infty} \frac{3^n - 2}{5^{2n}}$ is convergent or divergent. If it is convergent, find what is converges to.

Chapter 11 Map

```
11.1 -> 11.2 -> 11.5 -> 11.6
     |         |         |
     11.3    11.4    11.8
               |         |
               11.9    11.10 -> 11.11
```

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11.1 Sequences

Section Objective(s):

- Examine properties of sequences and their limits
- Use limits of functions to calculate limits of sequences

Definition(s) 11.1.1. A ____________ can be thought of as a list of numbers written in a definite order:

The number ____ is called the first term, ____ is the second term, and in general ____ is the nth term.

Remark 11.1.2. The sequence \{a_1, a_2, a_3, \ldots\} is also denoted by

Example 11.1.3. Write a function with domain \(\mathbb{Z}^+\) that defines the following list of numbers:

12, 14, 16, 18,...

Definition(s) 11.1.4. A sequence \(\{a_n\}\) has the ____________ and we write

if we can make the terms \(a_n\) as close to \(L\) as we like by taking \(n\) sufficiently large. If \(\lim_{n \to \infty} a_n \__________\), we say the sequence ______________ (or is ______________).

Otherwise, we say the sequences ______________ (or is ______________).
CHAPTER 11. INFINITE SEQUENCES AND SERIES

Theorem 11.1.5. If \( \lim_{x \to \infty} f(x) = L \) and \( f(n) = a_n \) when \( n \) is an integer, then

\[
\lim_{n \to \infty} a_n = L.
\]

Remark 11.1.6. Theorem [11.1.5] implies then that many of the properties of limits of functions apply to limits of sequences!

Theorem 11.1.7. If \( \lim_{n \to \infty} a_n = L \) and \( \lim_{n \to \infty} b_n = M \) then

\[
\lim_{n \to \infty} (a_n + b_n) = L + M, \quad \lim_{n \to \infty} (a_n \cdot b_n) = L \cdot M.
\]

Theorem 11.1.8. Squeeze Theorem for Sequences If \( a_n \leq b_n \leq c_n \) for \( n \geq n_0 \)

\[
\text{and } \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L, \text{ then } \lim_{n \to \infty} b_n = L.
\]

Theorem 11.1.9. If \( \lim_{n \to \infty} |a_n| = 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

Example 11.1.10. Calculate: \( \lim_{n \to \infty} \frac{\cos n}{n} \)

Remark 11.1.11. The sequence \( \{r^n\} \) is convergent if \(-1 < r \leq 1\) and divergent for all other values of \( r \).
11.1. SEQUENCES

**Definition(s) 11.1.12.** A sequence \( \{a_n\} \) is called ____________ if ____________ for all \( n \geq 1 \), that is \( a_1 < a_2 < a_3 \cdots \). It is called ____________ if ____________ for all \( n \geq 1 \).

A sequence is ____________ if it is either ____________ or ____________.

**Theorem 11.1.13.** If \( f(x) \) is an ____________ function and \( f(n) = a_n \) then \( \{a_n\} \) is an ____________ sequence. And similar for ____________.

**Example 11.1.14.** Show that the following sequence is decreasing: \( \left\{ \frac{1}{2^n} \right\} \)

**Example 11.1.15.** Consider the sequence: \( 1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \ldots \). Find a formula for \( a_n \).
Example 11.1.16. Consider the sequence \( \left\{ \frac{1 + 5n}{3 - 2n} \right\} \). Find the limit of the sequence.

Example 11.1.17. Consider the sequence \( \left\{ \left( \frac{-1}{5} \right)^n + \frac{\sin n}{n} \right\} \). Find the limit of the sequence.

Example 11.1.18. Consider the sequence \( \left\{ \frac{\ln 3n}{2n} \right\} \). Find the limit of the sequence.
11.2 Series

Section Objective(s):
- Define series and some different types of series
- Determine some strategies for identifying when series converge or diverge

Definition(s) 11.2.1. In general, if we try to add the terms of an infinite sequence \( \{a_n\}_{n=1}^{\infty} \) we get an expression of the form

\[
a_1 + a_2 + a_3 + \cdots + a_n + \cdots
\]

which is called an ____________________________ (or just a _________) and is denoted, for short by the symbol

Definition(s) 11.2.2. Given a series \( \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots \), let \( s_n \) denote its \( n \)th partial sum:

\[
\text{If the sequence } \{s_n\} \text{ is convergent and } \lim_{n \to \infty} s_n = s \text{ exists as a real number, then the series } \sum a_n \text{ is called } \text{________________________} \text{ and we write}
\]

The number \( s \) is called the ________ of the series. If the sequence \( \{s_n\} \) is divergent, then the series is called ________________.
Example 11.2.3. Determine if the following series converges or diverges: \(1 + 2 + 3 + 4 + 5 + \cdots\)

Example 11.2.4. Determine if the following series converges or diverges: \(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\)

**Definition(s) 11.2.5.** The previous series is called a __________________________. In general these are of the form

\[
a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n
\]

**Theorem 11.2.6.** The geometric series

is convergent if \(\left| r \right| < 1\) and its sum is

\[
\sum_{n=0}^{\infty} ar^n = a \left(1 - \left| r \right|\right)^{-1}
\]

If \(\left| r \right| \geq 1\), then the geometric series is divergent.
11.2. SERIES

**Theorem 11.2.7.** If the series \( \sum_{n=1}^{\infty} a_n \) is convergent, then 
\[
\lim_{n \to \infty} a_n = 0.
\]

**Theorem 11.2.8** (*nth term test for divergence*).

If \( \lim_{n \to \infty} a_n \) does not exist or if \( \lim_{n \to \infty} a_n \neq 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.

**Theorem 11.2.9.** If \( \sum a_n \) and \( \sum b_n \) are convergent series, then so are the series \( \sum ca_n \) (where \( c \) is a constant), \( \sum (a_n + b_n) \), and \( \sum (a_n - b_n) \), and

**Example 11.2.10.** Determine if the following series is convergent or divergent: 
\[
\sum_{n=1}^{\infty} \frac{-n}{2n + 5}
\]
Remark 11.2.11. If \( \lim_{n \to \infty} a_n = 0 \) it does not guarantee that the series \( \sum_{n=1}^{\infty} a_n \) is convergent.

Definition(s) 11.2.12. The harmonic series is defined as:

Theorem 11.2.13. The harmonic series is divergent

Example 11.2.14. Determine if the following series are convergent or divergent. If they are convergent, find what they converge to

(a) \( \sum_{n=0}^{\infty} \frac{3^n - 2}{5^{2n}} \)

(b) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n + 1}} \)
11.3 The Integral Test

Section Objective(s):
- Understand the hypotheses and conclusion of the Integral Test.
- Apply the Integral Test to some series

Theorem 11.3.1 (The Integral Test). Suppose \( a_n = f(n) \) and that \( f \) is

(i)

(ii)

(iii)

on \([1, \infty)\). If all these conditions are met then

(a) If \( \int_1^\infty f(x) \, dx \) is convergent, then \( \sum_{n=1}^{\infty} a_n \) is convergent.

(b) If \( \int_1^\infty f(x) \, dx \) is divergent, then \( \sum_{n=1}^{\infty} a_n \) is divergent.

Remark 11.3.2. Why the Integral Test makes sense
Example 11.3.3. Determine if \( \sum_{n=0}^{\infty} \frac{1}{n^2 + 1} \) converges or diverges.

**Theorem 11.3.4** (p-series test).

\[
\sum_{n=1}^{\infty} \frac{1}{n^p}
\]

is convergent if \( p > 1 \) and divergent if \( p \leq 1 \).

Example 11.3.5. Determine if the following series is convergent or divergent.

\[
\sum_{i=1}^{\infty} \frac{2}{1 + e^i}
\]
Theorem 11.3.6 (The math of increasing and decreasing positive continuous functions). Suppose $f_1(x)$ and $f_2(x)$ are increasing functions and $g_1(x)$ and $g_2(x)$ are decreasing functions on an interval $I$ then

- $f_1(x) + f_2(x)$ is _____________
- $f_1(x)f_2(x)$ is _____________
- $\frac{1}{f_1(x)}$ is _____________
- If $c > 0$ then
  - $cf_1(x)$ is _____________
  - $-cf_1(x)$ is _____________
- $g_1(x) + g_2(x)$ is _____________
- $g_1(x)g_2(x)$ is _____________
- $\frac{1}{g_1(x)}$ is _____________
- If $c > 0$ then
  - $cg_1(x)$ is _____________
  - $-cg_1(x)$ is _____________

Proof of one of these:
Example 11.3.7. Determine if the following series are convergent or divergent.

(a) \( \sum_{k=1}^{\infty} \frac{-3}{\sqrt{k^3}} \)

(b) \( \sum_{n=2}^{\infty} \frac{5}{n \ln n} \)
11.4 The Comparison Tests

Section Objective(s):
- Go over the Direct and Limit Comparison Tests
- Apply the Comparison tests to a variety of series problems

**Theorem 11.4.1 (The Direct Comparison Test).** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n$, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all $n$, then $\sum a_n$ is also divergent.

**Theorem 11.4.2 (The Limit Comparison Test).** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

where $c$ is a finite number and $c > 0$, then either both series converge or both diverge.

**Example 11.4.3.** Determine if the following series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{4}{3n - 1}$$
Example 11.4.4. Determine if the following series are convergent or divergent.

(a) \[ \sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1} \]

(b) \[ \sum_{n=1}^{\infty} \frac{1}{n^2 - n - 1} \]
Example 11.4.5. Determine if the following series are convergent or divergent.

(a) \[ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1} \]

(b) \[ \sum_{n=1}^{\infty} \frac{ne^{-n^2}}{3 + e^{-n}} \]
11.5 Alternating Series

Section Objective(s):

- Define Alternating Series
- Determine when Alternating Series converge and diverge.

Definition(s) 11.5.1. An alternating series is a series whose terms are alternately positive and negative. Here’s an example:

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \]

Remark 11.5.2. The \( n \)th term of an alternating series is of the form

where \( b_n \) is a positive number. (In fact, \( b_n = |a_n| \).)

Theorem 11.5.3 (Alternating Series Test). If the alternating series

\[ \sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \]

satisfies

(i)

(ii)

then the series is \( \underline{\text{convergent}} \).
Example 11.5.4. Determine if the following series are convergent or divergent.

(a) \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n \cdot 3^n} \)

(b) \( \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{1 + e^{2n}} \)
Example 11.5.5. Determine if the following series are convergent or divergent.

(a) \[ \sum_{n=1}^{\infty} (-1)^{n+1} (\sqrt{n+3} - \sqrt{n}) \]

(b) \[ \sum_{n=1}^{\infty} (-1)^{n} \cos \left( \frac{2\pi}{n} \right) \]
11.6 Absolute Convergence and the Ratio Test

Section Objective(s):
- Define absolutely and conditionally convergent.
- Understand the statement of the Ratio Test
- Apply the Ratio test to a variety of series

Definition(s) 11.6.1. A series $\sum a_n$ is called ____________________________ if the series of absolute values $\sum |a_n|$ is convergent.

Definition(s) 11.6.2. A series $\sum a_n$ is called ____________________________ if it is convergent but not absolutely convergent.

Remark 11.6.3. The series $\sum \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 11.6.4. If a series $\sum a_n$ is absolutely convergent then it is ____________________.

Example 11.6.5. Determine if the following series is absolutely or conditionally convergent: $\sum_{n=1}^{\infty} \frac{(-1)^n+1}{n^2}$
Theorem 11.6.6. The Ratio Test

(i) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \), then the series \( \sum_{n=1}^{\infty} a_n \) is ________________________________

(and therefore ________________________).

(ii) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1 \) or \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty \), then the series \( \sum_{n=1}^{\infty} a_n \) is ________________________.

(iii) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \), the Ratio Test ________________________;

that is, ________________________ can be drawn about the convergence or divergence of \( \sum a_n \).

Example 11.6.7. See what happens when you try to apply the ratio test to

(a) \( \sum_{n=1}^{\infty} \frac{1}{n} \)

(b) \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)

Remark 11.6.8. Recall that \( n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \)

Example: \( 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \)
**Example 11.6.9.** Determine if the following series is convergent or divergent: \[ \sum \frac{(2n)!}{(n!)^2} \]

**Example 11.6.10.** Determine if the following series is absolutely convergent, conditionally convergent, or divergent: \[ \sum \frac{(-1)^n n^2}{n!} \]
Example 11.6.11. Determine if the following series are absolutely convergent, conditionally convergent, or divergent:

(a) \( \sum n \left( \frac{3}{5} \right)^n \)

(b) \( \sum \frac{\cos(\pi n)}{3n} \)
11.8 Power Series

Section Objective(s):
- Define power series.
- Begin calculating closed forms of power series.
- Calculate radii and interval of convergence for power series.

Definition(s) 11.8.1. A \textit{power series} is a series of the form
\[ \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \]
where \( x \) is a variable and the \( c_n \)'s are constants called the coefficients of the series.

So now instead of a series approaching a number we can think that power series approach functions.

Additionally,

Definition(s) 11.8.2. A series of the form
\[ \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots \]
is called a \textit{power series in} \((x-a)\) or a \textit{power series centered at} \(a\) or a \textit{power series about} \(a\).

Example 11.8.3. Find a closed expression for the following series: \( 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \cdots \)
Theorem 11.8.4. For a given power series \( \sum_{n=0}^{\infty} c_n(x - a)^n \) there are only three possibilities:

(i)

(ii)

(iii)

Remark 11.8.5. The number \( R \) in case (iii) is called the radius of convergence of the power series. By convention, the radius of convergence is \( R = 0 \) in case (i) and \( R = \infty \) in case (ii). The interval of convergence of a power series is the interval that consists of all values of \( x \) for which the series converges.

Example 11.8.6. Find the open interval of convergence for the following power series:

\[ \sum_{n=0}^{\infty} \frac{n(x + 3)^n}{5^n} \]
Example 11.8.7. Find the radius and interval of convergence for the following power series: \[ \sum_{n=0}^{\infty} \frac{n(x + 3)^n}{n^2 + 1} \]

Example 11.8.8. Find the radius and interval of convergence for the following power series: \[ \sum_{n=1}^{\infty} x^n \ln n \]
Example 11.8.9. Find the radius and interval of convergence for the following power series: $\sum_{n=0}^{\infty} n! x^n$

Example 11.8.10. Find the interval of convergence for the following power series: $\sum_{n=1}^{\infty} (-1)^n (3x + 5)^n$
11.9 Representations of Functions as Power Series

Section Objective(s):

- Understand how functions can be represented as power series
- Learn differentiation and integration for power series

Theorem 11.9.1.

Example 11.9.2. Express \( \frac{1}{1 - 3x^2} \) as a power series. Find the radius of the convergence.

Example 11.9.3. Express \( \frac{x}{3 + x} \) as a power series. Find the interval of the convergence.
Theorem 11.9.4. If the power series $\sum c_n(x - a)^n$ has radius of convergence $R > 0$, then the function $f$ defined by

is differentiable (and therefore continuous) on the interval $\left(a - R, a + R\right)$ and

(i) $f'(x) =

(ii) $\int f(x) \, dx =$

The radii of convergence of the power series in Equations (i) and (ii) are both $R$.

Example 11.9.5. Find a power series representation for $\frac{1}{(1 - x)^2}$ and its radius of convergence.
Example 11.9.6. Find a power series representation for $\ln(1 + x)$ and its radius of convergence.

Example 11.9.7. Find a power series representation for $\tan^{-1}(1 + x)$ and its radius of convergence.
11.10 Taylor and Maclaurin Series

Let's consider the function \( f(x) = 1/x \) and polynomials that approximate \( f(x) \) as best they can at \( a = 1 \).

The best degree 0 polynomial (which we will call \( T_0(x) \)) can agree with \( f \) at 1. That is

The best degree 1 polynomial (which we will call \( T_1(x) \)) can agree with \( f \) at 1 and have the same slope. We should remember this from Calculus 1 as the ____________

The best degree 2 polynomial (which we will call \( T_2(x) \)) can agree with \( f \) at 1, have the same slope, and the same concavity. That is
Theorem 11.10.1. If $f$ has a power series representation (expansion) at $a$, that is, if

$$f(x) =$$

then its coefficients are given by the formula

$$c_n =$$

Definition(s) 11.10.2. If $f$ has a power series expansion at $a$, then it must be of the following form:

$$f(x) =$$

This is called the ________________ (or about $a$ or centered at $a$). For the special case $a = 0$ the Taylor series becomes

$$f(x) =$$

This case arises frequently enough that it is given the special name ____________________.

Example 11.10.3. Find the $2^{nd}$ degree Maclaurin polynomial for $f(x) = \sqrt{x + 1}$. 
**Definition(s) 11.10.4.** In the case of the Taylor series the partial sums are

\[ T_n(x) = \sum_{i=0}^{n} f^{(i)}(a) \frac{(x-a)^i}{i!} \]

Notice that \( T_n \) is \( \text{polynomial} \) of degree \( n \) called the \( \text{degree } n \text{ Taylor polynomial of } f \text{ at } a \). In general, \( f(x) \) is the sum of its Taylor series if

\[ f(s) = \]

**Definition(s) 11.10.5.** If we let

\[ R_n(x) = f(x) - T_n(x) \]

so that

Then \( R_n \) is called the \( \text{remainder} \) of the Taylor series.

**Example 11.10.6.** Find the Taylor series generated by \( f(x) = \frac{1}{x} \) centered at \( a = 2 \).
Theorem 11.10.7 (Important Maclaurin Series and their Radii of Convergence).

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = R = 1
\]

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = R = \infty
\]

\[
sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = R = \infty
\]

\[
cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = R = \infty
\]

\[
\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = R = 1
\]

\[
\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = R = 1
\]

\[
(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = R = 1
\]

**Proof:** \(e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\) and that \(R = \infty\)
Example 11.10.8. Consider \( f(x) = e^{4x} \)

(a) Find the \( n^{th} \) term Maclaurin polynomial for \( f \) using Definition 11.10.4.

(b) Find the \( n^{th} \) term Maclaurin polynomial for \( f \) using Theorem 11.10.7.
Theorem 11.10.9. If \( f(x) = T_n(x) + R_n(x) \), where \( T_n \) is the \( n \)th degree polynomial of \( f \) at \( a \) and for \( |x - a| < R \), then \( f \) is equal to the sum of its Taylor series on the interval \( |x - a| < R \).

Theorem 11.10.10. Taylor Remainder Theorem Suppose \( f(x) = T_n(x) + R_n(x) \) where \( T_n \) is the \( n \)th degree Taylor polynomial of \( f \) at \( a \). Then \( R_n \) is the Lagrange form of the remainder of order \( n \) and is given by

\[
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}
\]

for some \( c \) between \( a \) and \( x \).

Example 11.10.11. Find the Taylor polynomial of degree 3 for the function \( f(x) = \sqrt{x + 9} \) about the point \( x = -5 \). Then write \( R_3(x) \) as a function of \( x \) and \( c \).
11.11 Additional Applications and Problems for Taylor Polynomials

Section Objective(s):

- Use Taylor polynomials to calculate terms for various binomial series
- Multiply known Taylor polynomials to find the terms of Taylor polynomials for additional functions
- Use Taylor polynomials to help evaluate limits
- Use Taylor polynomials to determine what a series converges to.
- State and apply the Taylor Inequality

Definition(s) 11.11.1. A generalized binomial coefficient can be defined as:

\[
\binom{\alpha}{0} = 1 \\
\binom{\alpha}{1} = \frac{\alpha}{1!} = \alpha \\
\binom{\alpha}{2} = \frac{\alpha(\alpha - 1)}{2!} = \frac{1}{2} \alpha(\alpha - 1) \\
\vdots \\
\binom{\alpha}{n} = \\
\]

Example 11.11.2. Find the first four terms of the binomial series for the following function:

\[f(x) = (1 + x^3)^{-1/2}\]
11.11. ADDITIONAL APPLICATIONS AND PROBLEMS FOR TAYLOR POLYNOMIALS

Example 11.11.3. Find the Maclaurin series for the function $f(x) = \sqrt{1 + x}$ and find its radius of convergence.

Example 11.11.4. Find the first 3 terms for the Maclaurin series for the function $f(x) = e^{2x} \sin(x)$

Example 11.11.5. Find the first 4 terms for the Maclaurin series for the function $f(x) = \frac{x}{\cos(x)}$
Example 11.11.6. Evaluate \( \lim_{x \to 0} \frac{\sin(x^3) - x^3}{x^9} \)

Example 11.11.7. Evaluate \( \lim_{x \to 0} \frac{x - \ln(x + 1)}{x^2} \)
Example 11.11.8. Use Taylor series to evaluate \( \sum_{k=0}^{\infty} \frac{2^k}{k!} \)

Example 11.11.9. Use Taylor series to evaluate \( \frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \cdots \)
Theorem 11.11.10. Taylor’s Inequality If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq M \frac{(n+1)!}{d^{n+1}} |x - a|^{n+1}$$

Example 11.11.11. Consider approximating $f(x) = e^{2x}$ with $T_2(x) = 1 + 2x + 2x^2$. Find the maximum error by making this approximation on the interval $[-2, 2]$. 
Chapter 10

Parametric Equations and Polar Coordinates
10.1 Curves Defined by Parametric equations

Section Objective(s):
- Define parametric equations.
- Use parametric equations to sketch curves.
- Transform between parametric and Cartesian equations.

Definition(s) 10.1.1. Suppose that \( x \) and \( y \) are both given as functions of a third variable \( t \) (called a parameter) by the equations

\[
\begin{align*}
x &= f(t) \\
y &= g(t)
\end{align*}
\]

(called 

Each value of \( t \) determines a point \( (x, y) \), which we can plot in a coordinate plane. As \( t \) varies, the point \( (x, y) = (f(t), g(t)) \) varies and traces out a curve \( C \), which we call a

Example 10.1.2. Plot the parametric equations \( x = 2t \) and \( y = 3 - t \). Find the corresponding Cartesian equation

Remark 10.1.3. Sometimes we restrict \( t \) to lie in a finite interval. In general, the curve with parametric equations

\[
\begin{align*}
\text{has } (f(a), g(a)) \text{ and } (f(b), g(b))
\end{align*}
\]
Example 10.1.4. Plot the parametric equations $x = 2 \sin t$ and $y = 2 \cos t$ for $t \in [0, \pi/2]$. Find the corresponding Cartesian equation.

Example 10.1.5. Plot the parametric equations $x = 3 \sin 2t$ and $y = -5 \cos 2t$ for $t \in [0, \pi/2]$. Find the corresponding Cartesian equation.
Example 10.1.6. Find the corresponding Cartesian equation for the parametric equations $x = 2 \sec(3t)$ and $y = -5 \tan(3t)$.

Example 10.1.7. Find the a parametric equation for $y = x^2 - x + 3$ which starts at (0, 3) when $t = 0$ and ends a (3, 9) when $t = 5$. 
10.2 Calculus with Parametric Curves

Section Objective(s):
- Apply methods of calculus to parametric curves
- Practice solving problems involving tangents and arc length

Definition(s) 10.2.1. Suppose the curve is traced out once by the parametric equations
\[ x = f(t) \] and \[ y = g(t) \] and that \( f \) and \( g \) are differentiable functions. Assuming we want to find the tangent line at a point on the curve where \( y \) is also a differentiable function of \( x \) then the Chain Rule gives

\[ \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \]

If \( \frac{dx}{dt} \neq 0 \), we can solve for \( \frac{dy}{dx} \):

Example 10.2.2. Find the equation of the tangent line to the curve given by:

\[ x = 3t - \sin(t), \quad y = 1 - \cos(t), \quad \text{at } t = \pi/3 \]
Example 10.2.3. Find an equation of the tangent to the curve \( x = 1 + \ln t, \; y = t^2 + 3 \) at the point \((1, 4)\).

Example 10.2.4. Find the points on the curve \( x = t^3 - 3t, \; y = t^2 - 3 \) where the tangent is horizontal or vertical.
Theorem 10.2.5. If a curve $C$ is described by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where $f'$ and $g'$ are continuous on $[\alpha, \beta]$ and $C$ is traversed exactly once at $t$ increases from $\alpha$ to $\beta$, then the length of $C$ is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Example 10.2.6. Find the length of a circle of radius 3 defined by: $x = 3 \cos(t), y = 3 \sin(t)$ on $0 \leq t \leq 2\pi$
Example 10.2.7. Find the exact length of the curve $x = 1 + 3t^2$, $y = 2 + 2t^3$ for $t \in [0, 1]$

Example 10.2.8. Find the length of the curve $x = e^t + e^{-t}$, $y = 1 - 2t$ for $t \in [0, 2]$
10.3 Polar Coordinates

Section Objective(s):

- Understand the concept of polar coordinates
- Transfer back and forth between Cartesian and polar coordinates.
- Graph equations given in polar coordinates.
- Find tangent lines given in polar coordinates.

What are polar coordinates?

Theorem 10.3.1.

Example 10.3.2. Graph the polar coordinates

(a) \((\theta, r) = (7\pi/6, 2)\)

(b) \((\theta, r) = (-\pi/3, -3)\)

(c) \((\theta, r) = (3\pi, 0)\)
Example 10.3.3. Convert the following polar coordinates into Cartesian coordinates

(a) \((\theta, r) = (0, 2)\)

(b) \((\theta, r) = (-\pi/4, 5)\)

Example 10.3.4. Convert the following Cartesian coordinates into polar coordinates

(a) \((x, y) = (0, 3)\)

(b) \((x, y) = (-2, 2)\)

Definition(s) 10.3.5. The graph of a polar equation \(r = f(\theta)\), or more generally \(F(r, \theta) = 0\), consists of all points \(P\) that have at least one polar representation \((r, \theta)\) whose coordinates satisfy the equation.
Example 10.3.6. Consider the curve given by \( r = 2 \sin \theta \)

(a) Graph the curve using the point plotting method

(b) What geometric figure is the curve?

(c) Transform the curve into Cartesian coordinates to confirm your suspicions in part (b)
Example 10.3.7. Consider the curve given by $\frac{1}{r} = \sin \theta + \cos \theta$ for $\theta \in (-\pi/4, 3\pi/4)$

(a) Graph the curve using the point plotting method

(b) What geometric figure is the curve?

(c) Transform the curve into Cartesian coordinates to confirm your suspicions in part (b)
**Definition(s) 10.3.8.** To find a tangent line to a polar curve \( r = f(\theta) \), we regard \( \theta \) as a parameter and write its parametric equations as

\[
\begin{align*}
x &= r \cos \theta = f(\theta) \cos \theta \\
y &= r \sin \theta = f(\theta) \sin \theta
\end{align*}
\]

Then using the method of finding slopes of parametric curves and the Product Rule, we have

**Example 10.3.9.** Find the slope of the tangent line to the curve \( r = 1 + 2 \cos \theta \) at \( \theta = \pi/6 \).
10.4 Areas and Lengths in Polar Coordinates

Section Objective(s):

- Apply the formula for the area of a region whose boundary is given by polar coordinates
- Apply the formula for the arc length of a curve given by polar coordinates

Theorem 10.4.1. Suppose the boundary of a region $R$ is given by the polar equation $r = f(\theta)$. The area $A$ of $R$ is given by

or

Example 10.4.2. Find the area enclosed by the cardioid $r = 2(1 + \cos(\theta))$.

Remark 10.4.3. Don’t forget, symmetry is your friend.
A quick lesson in intersecting functions

Example 10.4.4. Consider the functions \( r = 2 \cos \theta \) and \( r = 2 \sin \theta \) for \( \theta \in [0, \pi] \).

(a) Graph both of the functions together on the graph below.

(b) How would the picture look if you graphed these curves with \( \theta \in [0, 2\pi] \)?

(c) Find all \((r, \theta)\) values were these two functions intersect on.

(d) Find all \((x, y)\) values were these two functions intersect on.

(e) Calculate the area shared by these two circles.
Theorem 10.4.5. Suppose \( f(\theta) \) and \( g(\theta) \) are continuous polar functions and that \( f(\theta) \geq g(\theta) \geq 0 \) for all \( \theta \in [\alpha, \beta] \). The area bounded between \( f(\theta) \) and \( g(\theta) \) on \([\alpha, \beta]\) is given by

\[
A = \int_{\alpha}^{\beta} \frac{1}{2} \left[ (f(\theta))^2 - (g(\theta))^2 \right] d\theta
\]

Example 10.4.6. Find the area inside the cardioid \( r = 1 + \cos \theta \) and outside the circle \( r = 3/2 \) (shown below)
Theorem 10.4.7. The length of a curve with polar equation \( r = f(\theta), a \leq \theta \leq b \), is

\[
L = \int_{a}^{b} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta
\]

Example 10.4.8. Find the exact length of the polar curve \( r = 2 \cos \theta \) for \( 0 \leq \theta \leq \frac{3\pi}{4} \)

Example 10.4.9. Find the exact length of the polar curve \( r = \theta^2 \) for \( 0 \leq \theta \leq \pi \)