

Vectors in Space

Suppose $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$:

- **Unit Vectors:**

$$\mathbf{i} = \langle 1, 0, 0 \rangle$$

$$\mathbf{j} = \langle 0, 1, 0 \rangle$$

$$\mathbf{k} = \langle 0, 0, 1 \rangle$$

- **Length** of vector \mathbf{u}

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

- **Dot Product:**

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= |\mathbf{u}||\mathbf{v}| \cos \theta\end{aligned}$$

- **Cross Product:**

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- **Vector Projection:** $\text{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2} \mathbf{u}$

Partial Derivatives

- **Chain Rule:** Suppose $z = f(x, y)$ and $x = g(t)$ and $y = h(t)$ are all differentiable then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Curves and Planes in Space

- **Line** parallel to \mathbf{v} : $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$

- **Plane** normal to $\mathbf{n} = \langle a, b, c \rangle$:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- **Arc Length** of curve $\mathbf{r}(t)$ for $t \in [a, b]$.

$$L = \int_a^b |\mathbf{r}'(t)| dt$$

- **Unit Tangent Vector** of curve $\mathbf{r}(t)$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

More on Surfaces

- **Directional Derivative:** $D_{\mathbf{u}}f(x, y) = \nabla f \cdot \mathbf{u}$

- **Second Derivative Test** Suppose

$f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let

$$D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- If $D < 0$ then $f(a, b)$ is a saddle point.

Geometry / Trigonometry

- Area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $A = \pi ab$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2 \sin x \cos x$

Multiple Integrals

- **Area:** $A(D) = \iint_D 1 \, dA$
- **Volume:** $V(E) = \iiint_E 1 \, dV$

Polar/Cylindrical

- Transformations

$$\begin{aligned}r^2 &= x^2 + y^2 \\x &= r \cos \theta \\y &= r \sin \theta \\y/x &= \tan \theta\end{aligned}$$

- $\iint_D f(x, y) \, dA = \iint_D f(r \cos \theta, r \sin \theta) r \, dr \, d\theta$
- $\iiint_E f(x, y, z) \, dV = \iiint_E f(r \cos \theta, r \sin \theta, z) r \, dz \, dr \, d\theta$

Spherical

- Transformations

$$\begin{aligned}x &= \rho \sin \phi \cos \theta \\y &= \rho \sin \phi \sin \theta \\z &= \rho \cos \phi \\\rho^2 &= x^2 + y^2 + z^2\end{aligned}$$

- $\iiint_E f(x, y, z) \, dV = \iiint_E f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta$

Additional Definitions

- $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$
- $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$
- \mathbf{F} is conservative if $\text{curl}(\mathbf{F}) = 0$

Line Integrals

- **Fundamental Theorem of Line Integrals**

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- **Green's Theorem**

$$\int_C P \, dx + Q \, dy = \iint_D (Q_x - P_y) \, dA$$