Vectors in Space

Suppose $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$:

- **Unit Vectors**:
$$\mathbf{i} = \langle 1, 0, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1, 0 \rangle$$
$$\mathbf{k} = \langle 0, 0, 1 \rangle$$
- **Length** of vector $\mathbf{u}$
$$|\mathbf{u}| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}$$
- **Dot Product**:
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = |\mathbf{u}||\mathbf{v}| \cos\theta$$
- **Cross Product**:
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
- **Vector Projection**: $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2}\, \mathbf{u}$

Partial Derivatives

- **Chain Rule**: Suppose $z = f(x, y)$ and $x = g(t)$ and $y = h(t)$ are all differentiable then
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Curves and Planes in Space

- **Line** parallel to $\mathbf{v}$: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$
- **Plane** normal to $\mathbf{n} = \langle a, b, c \rangle$:
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
- **Arc Length** of curve $\mathbf{r}(t)$ for $t \in [a, b]$.
$$L = \int_a^b |\mathbf{r}'(t)|\, dt$$
- **Unit Tangent Vector** of curve $\mathbf{r}(t)$
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

More on Surfaces

- **Directional Derivative:** $D_{\mathbf{u}} f(x, y) = \nabla f \cdot \mathbf{u}$
- **Second Derivative Test** Suppose $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let
$$D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$
  (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
  (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
  (c) If $D < 0$ then $f(a, b)$ is a saddle point.

Geometry / Trigonometry

- Area of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $A = \pi ab$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2 \sin x \cos x$

Multiple Integrals

- **Area**: $A(D) = \iint_D 1 \ dA$
- **Volume**: $V(E) = \iiint_E 1 \ dV$

Polar/Cylindrical

- Transformations

$$\begin{aligned} r^2 &= x^2 + y^2 \\ x &= r\cos\theta \\ y &= r\sin\theta \\ y/x &= \tan\theta \end{aligned}$$

- $\iint_D f(x,y) \ dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$
- $\iiint_E f(x,y,z) \ dV =$
  $\iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$

Spherical

- Transformations

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \\ \rho^2 &= x^2 + y^2 + z^2 \end{aligned}$$

- $\iiint_E f(x,y,z) \ dV =$
  $\iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta$

Additional Definitions

- $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$
- $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$
- $\mathbf{F}$ is conservative if $\text{curl}(\mathbf{F}) = 0$

Line Integrals

- **Fundamental Theorem of Line Integrals**

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- **Green's Theorem (Tangential Form)**

$$\int_C P \ dx + Q \ dy = \iint_D (Q_x - P_y)\, dA$$

- **Green's Theorem (Normal Form)**

$$\int_C P \ dy - Q \ dx = \iint_D (P_x + Q_y)\, dA$$

Integrals over Surfaces

- **Stokes' Theorem**

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$$

- **Divergence Theorem**

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div}\,\mathbf{F}\, dV$$