Introduction to Modeling

Review of Section 1.2, 1.3, 1.4, 1.5, 1.6

Objectives

- Students should be able to create mathematical models from a given description of a physical sitation.
- Conversely, given the differential equations (for example, of a population model), students should know the meaning of the equation coefficients and then be able to describe what type of population system is described by these equations.
- Students should be able to obtain a qualitative description of the solutions of autonomous differential equations.

Monday, September 9th

(1) (Basic Concepts, and Section 1.5 Linear Equation)

Consider the initial value problems of function $y = y(t)$.

⟨1⟩

$$y' = \frac{t}{y + t^2 y}, \qquad y(0) = 1.$$

⟨2⟩

$$y' = -3(y-5)(y-10)(y-20), \qquad y(0) = 6, \qquad 5 < y < 10.$$

⟨3⟩

$$y' = \frac{2y^2 - 10t^2}{ty}, \qquad y(1) = 3, \qquad 0 < t < \sqrt{10}.$$

⟨4⟩

$$y' = \frac{2y - 9t}{t}, \qquad y(1) = 2, \qquad t > 0.$$

(1a) Find **autonomous equation(s)** with the form $y' = f(y)$, if any.

(1b) Find **separable equation(s)**, or equation(s) that could be transformed to a separable equation, if any, and transform it to separable form $h(y)y' = g(t)$.

(1c) Find **Euler Homogeneous Equation(s)**, if any, and rewrite it to the form $y' = F(y/t)$.
Hint: Try to follow textbook example 1.4.13. Recall the definition of **homogeneous of degree n**.

(1d) Find **linear equation(s)**, if any, and rewrite it to the form $y' = a(t)y + b(t)$. Choose one of the **linear** equation(s) you find, and find the **explicit** expression of the solution $y(t)$.
Hint: If the linear equation has the form $y' = a(t)y + b(t)$, the **integrating factor** $\mu(t)$ is

$$\mu(t) = e^{A(t)}, \text{ where } A(t) \text{ is a solution of } A'(t) = -a(t).$$

Wednesday, September 11th

Recall what we found on Monday.

(2) (Section 1.4, Separable Equation)

Consider the initial value problem

$$y' = \frac{t}{y + t^2 y}, \qquad y(0) = 1.$$

(1a) Find an **implicit** expression for all the solutions, $y(t)$, of the differential equation above, in the form $\psi(t, y) = c$, where c collects all constant terms.

(1b) Find the **explicit** expression of the solution $y(t)$ of the initial value problem above.

(3) (Section 1.6, Example of ODE with Non-uniqueness)

Consider initial value problem

$$y' = y^{2/3}, y(0) = 0.$$

We note that $y(t) = 0$ is a solution. Try to find other solution(s).

Hint: Use the method you solve separable equation to find the solution.

(4) (Section 1.4, Euler Homogeneous Equation)

Consider the initial value problem

$$y' = \frac{2y^2 - 10t^2}{ty}, \qquad y(1) = 3, \qquad 0 < t < \sqrt{10}.$$

(2a) Find an **implicit** expression for all the solutions, $y(t)$, of the differential equation above, in the form $\psi(t, y) = c$, where c collects all constant terms.
Hint: You should recall what you found in Problem **(1c)**.

(2b) Find the **explicit** expression of the solution $y(t)$ of the initial value problem above.
Hint: Recall that $\int \frac{1}{t} dt = \ln(|t|) + C$ with C constant. Don't forget the absolute value.

(5) (Section 1.3, Qualitative Analysis)

Consider the population model

$$P' = -3\left(P - 5\right)\left(P - 10\right)\left(P - 20\right),$$

where $P = P(t)$ is the population at time t.

(4a) Determine the **equilibria** of the differential equation.

(4b) Determine the intervals where the population is **increasing** and where it is **decreasing**.
Hint: When a function is **increasing**, the derivative of the function is **positive**. What about the case when a function is **decreasing**?

(4c) Determine the stability of the equilibria. Plot the phase diagram on $P' - P$ plane.

(4d) Sketch all of the **equilibria**, and the solution when $P \in (5, 10)$, on $P - t$ plane.
Hint: You should sketch the equilibria first, which will give you some idea of the remaining part. If you are still confused, try to sketch a solution with initial value $P(0) = 6$.

Wednesday, September 11th, Discussion

(1) (Section 1.2, Exponential Model and Immigration Model)

A radioactive material has a **decay rate** proportional to the amount of radioactive material present at that time, with a proportionality factor of 5 per unit time.

- **(3a)** Write a **differential equation** of the form $P' = F(P)$, which models this situation, where P is the amount of radioactive material micrograms as function of time.
- **(3b)** Now, assume that we have the same radioactive material decaying as above, but we are **adding** additional material (of the same type) **at a constant rate of 3** micrograms per unit time. Write the differential equation in this case.
- **(3c)** Are you able to solve the ordinary differential equation? If so, briefly describe the method you will use. Assuming the **initial** amount of radioactive material is 10 micrograms, is the solution **unique**? Discuss with your friend **without** actually solving it.

(2) (Section 1.2, Interacting Species Model)

Consider two populations of organisms, which occupy the same environment. Let $A(t)$ and $B(t)$ denote the number of organisms at time t of the first and second kind, respectively. They are modeled by the following system of equations

$$A' = 2A - \frac{A^2}{10} - 3AB$$
$$B' = 3B - \frac{B^2}{5} + 2AB.$$

- **(5a)** Determine the **growth rate coefficients** of A and B (when the organisms are not sensing that resources are limited).
- **(5b)** Determine the **carrying capacity** of each of the populations (if the other population didn't exist).
- **(5c)** Determine the interaction between the two populations: are they cooperating, are they competing? If they have different types of behaviors, which one is **helping** and which one is **hurting** the other?