Second Order Linear Differential Equation

Linearity. Superposition. General Solution.

Objectives

- **Basic concepts:** order, homogeneous and non-homogeneous, constant or variable coefficients, linear differential equation, linear operator, linear dependent and linear independent, linear combination, fundamental solutions, general solution, particular solution
- **Important Properties:** Existence and Uniqueness; Superposition; Formulation of fundamental solutions and general solution
- **Theory of Solutions of Linear Homogeneous/Non-homogeneous Differential Equation:** Students should refer to Section 2.1 and Section 2.3.1 of the textbook.
- **Other Material in Chapter 2:** Solve homogeneous equation (2.2); Solve non-homogeneous equation (2.3); Models: Mass-Spring System (2.1), Kirchhoff's voltage law (2.3).

Example 1 (Linear)

1. Assume $L(y(t)) = y''(t) - 3y(t)$, compute $L(t^3)$.
2. **(Theorem 2.1.4)** For fixed continuous functions $a_1(t), a_2(t)$, show the operator $L(y(t)) = y''(t) + a_1(t)y'(t) + a_2(t)y(t)$ is a linear operator.
3. **(Example 2.1.11)** Let $y_1(t) = \sin(t), y_2(t) = 2\sin(t), y_3(t) = t\sin(t), y_4(t) = (2t+3)\sin(t)$, and show that, **(1)** y_1 and y_2 are linear dependent; **(2)** y_1 and y_3 are linear independent; **(3)** y_4 is linear combination of y_1 and y_3.

Exercise 1

1. Define the operator L, such that $L(y(t)) = y'''(t) + 3y(t)$. Compute $L(t^4)$. Show that L is linear.
2. Let $y_1(t) = t+1,\ y_2(t) = t^2$. Determine whether they are linear dependent or not. Write down the general formula of linear combination of y_1 and y_2.

Example 2.1 (Homogeneous Equation) Consider the differential equation

$$t^2 y'' - 2y = 0, \qquad t > 0.$$

Define functions

$$y_1(t) = t^2, \qquad y_2(t) = \frac{1}{t}, \qquad y_3(t) = 2t^2 + \frac{1}{t}.$$

1. Is the given equation **linear**? Is the given equation **homogeneous**? What is the **order** of the differential equation?

2. Find a set of **fundamental solutions** of the equation.

   Hint: Verify the two functions y_1, y_2 are **linear independent** solutions of the equation.

3. Find **general solution** of the equation.

4. Based on the general solution you find, determine whether function $y_3(t)$ is a solution of the equation.

5. Find another set of **fundamental solutions** of the equation, and use this result to write the **general solution** in a different way.

Example 2.2 (Non-homogeneous Equation) Consider the differential equation

$$t^2 y'' - 2y = 2t, \qquad t > 0.$$

1. Is the given equation **linear**? Is the given equation **homogeneous**? What is the **order** of the differential equation?

2. Verify function $y_4(t) = -t$ is a solution of the equation.

3. Find a **particular solution** of the equation.

4. Find **general solution** of the equation.

5. Find another **particular solution** of the equation that is different from what we find previously.

Exercise 2 Consider a second order **homogeneous** differential equation

$$(t+1)y'' + (t-1)y' - 2y = 0, \quad t > -1,$$

and a **non-homogeneous** one

$$(t+1)y'' + (t-1)y' - 2y = t+1, \quad t > -1.$$

Define functions

$$y_1(t) = 1 + t^2, \qquad y_2(t) = e^{-t}, \qquad y_3(t) = t^2 - t + 1.$$

1. Verify that functions y_1, y_2 are solutions of the **homogeneous** equation.

2. Verify that function y_3 is solution of the **non-homogeneous** equation.

3. Find a set of **fundamental solutions** of the **homogeneous** equation.

4. Find **general solution** of the **homogeneous** equation.

5. Find a **particular solution** of the **non-homogeneous** equation.

6. Find **general solution** of the **non-homogeneous** equation.

7. Use the proper general solution, to determine whether function $y_4(t) = t^2 - 2t + 1$ is a solution of the **non-homogeneous** equation.

8. Solve the **homogeneous** equation with initial condition $y(0) = 1$. Do you have unique solution in this case? Why or Why not?