

The Eight Annual Herzog Prize Examination

November 8, 1980

Problem 1: Find all positive integral solutions of

$$x^Y = y^{x-Y}$$

Problem 2: (T. Butts) Prove that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \int_0^{\infty} \frac{\sin^2 x}{x^2} dx .$$

Problem 3: (T. Butts) If k different integers are chosen at random from the set $\{1, 2, 3, \dots, p\}$ where p is prime, find the probability that their sum is divisible by p .

Problem 4: (L.M. Kelly) Two wire tracks follow along the curves $y = 1/x$ and $y = -1/x$, $x > 0$. A sphere of radius 1 rolls toward the origin along these tracks starting at a great distance from the origin. Describe the path followed by the center of the sphere.

Problem 5: (M.A.A. E 1336) Let $f(0) > 0$, $f(1) < 0$. Suppose that there is a continuous function g such that $f+g$ is non-decreasing on $[0,1]$. Show that $f(x) = 0$ for some $0 < x < 1$.

Problem 6: (L.M. Kelly) Is there a polyhedron with exactly 10 faces, each a pentagon?