THE THIRTEENTH ANNUAL HERZOG PRIZE EXAMINATION

November 9, 1985

Problem 1: Suppose

$$S = f(1) + f(2) + f(3) + \cdots$$

where $f(mn) = f(m)f(n) > 0$ for all m and n. Compute

$$f(1) + f(3) + f(5) + f(7) + \cdots ,$$

$$f(2) + f(4) + f(6) + f(8) + \cdots ,$$

and

$$f(1) - f(2) + f(3) - f(4) + f(5) - \cdots .$$

Problem 2: (M.J. Winter) What is the expected number of throws of a fair coin until heads occur twice in succession.

Problem 3: (J.G. Hocking) Given two circles and a point P on one, construct a circle through P tangent to both given circles.

Problem 4: (Wade C. Ramey) Let $P(x,y)$ be a polynomial where

$$\iint_D P(x,y)\,dxdy = 0$$

over every disk D of radius 1 containing the origin. Prove that $P(x,y)$ is the 0 polynomial.

Problem 5: (L.M. Kelly) Is the volume of a tetrahedron a function of the areas of its (four) faces?

Problem 6: (J.G. Hocking) Suppose f is differentiable on $[a,b]$ with $f(a) = a$ and $f(b) = b$. Do there exist $a < x_1 < x_2 < b$ with

$$\frac{1}{f'(x_1)} + \frac{1}{f'(x_2)} = 2 \text{ ?}$$