

Michigan State University
Herzog Mathematical Competition
November 7, 1987

1. Prove that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not an integer for any n .

Hills

2. If $\{a_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} e^{-a_n}$ diverges, show that $\sum_{n=1}^{\infty} \frac{1}{a_n^k}$ also diverges for any fixed positive integer k .

Asic

3. Consider all differentiable functions, $f(x)$, defined for all real x such that $f(0) = f'(0) = 1$ and $|f'(x) - 1| \leq |x|$.

By continuity each such function is nonnegative in a small interval around $x = 0$. Prove even more: there is an interval in which all the functions are nonnegative and find the length of the largest such interval.

Herzog

4. Let $G = \{0, 1, 3, 4, 9, \dots\}$ be the set of all those nonnegative integers which when written in base 3 have no digits equal to

2. Prove

(a) no three numbers in G are in arithmetic progression;

(b) if b is any positive integer not in G then there exists two numbers in G , a_1 and a_2 with $a_1 < a_2 < b$ and a_1, a_2, b are in arithmetic progression.

5. $S = \{P_1, P_2, \dots, P_n\}$ is a set of n points in the plane with no two distances $d(P_i, P_j)$ equal. Each point is joined by a segment to its nearest neighbor.

Prove: (a) no subset of these segments forms a closed polygon; (b) the number of these segments with a common endpoint is at most 5. Can 5 be attained?

- Sonneborn 6. Five professors attending a seminar find it hard to stay awake. On one occasion
- (a) each professor fell asleep exactly twice during the lecture and
 - (b) each two professors were simultaneously asleep at least once.

Prove that at least three must have been simultaneously asleep at some time during the lecture.