THE SIXTEENTH HERZOG PRIZE EXAMINATION

November 5, 1988

Problem 1

Determine (with proof) which is larger, $\left(\frac{21}{20}\right)^{100}$ or 121.

Problem 2

Determine (with proof) whether an equilateral triangle can be placed in the plane so that the coordinates of all three vertices are integers.

Problem 3

Call a subset of $\{1,2,\ldots,n\}$ unfriendly if it contains no two consecutive integers. Prove that the number of unfriendly subsets of $\{1,2,\ldots,n\}$ is the $(n+1)^{st}$ Fibonacci number F_{n+1}. (Here the empty set counts as an unfriendly set. The Fibonacci numbers are defined by the recursion $F_{n+1} = F_n + F_{n-1}$ with $F_0 = F_1 = 1$.)

Problem 4

Determine (with proof) for which integers n

$$f(n) = n(n+1)(n+2)(n+3) + 1$$

is a perfect square.

Problem 5

Prove that the series

$$\sum_{n=0}^{\infty} (3x - x^2)\left(\frac{2x^2 - 5x + 2}{x+2}\right)^n$$

converges for x in the interval $[0,3]$. Let $f(x)$ be the sum of this series. Sketch the graph of f over $[0,3]$.

Problem 6

Let $v_1, v_2, \ldots, v_n$ be the (consecutive listed) vertices of a regular polygon inscribed in the unit circle. Prove that

$$|v_1v_2| \cdot |v_1v_3| \cdot |v_1v_4| \cdot \ldots \cdot |v_1v_n| = n,$$

where $|v_1v_i|$ denotes the distance from v_1 to v_i.