THE TWENTY-SECOND HERZOG PRIZE EXAMINATION

November 5, 1994

1. Find the sum of all the five-digit numbers that can be formed by arranging all of the five digits 1, 2, 3, 4, 5.

2. Let $f(x) = x^3 - 3x + 2$. Suppose that the cubic curve $y = f(x)$ is cut by a non-vertical straight line in the points $A(a, f(a))$, $B(b, f(b))$ and $C(c, f(c))$. Find (with proof) the coordinates of C in terms of a and b.

3. Find (with proof) the smallest constant K such that
$$(x+y)^3 \leq K(x^3 + y^3)$$
for all positive numbers x, y.

4. Prove: For every positive integer n, there is a positive integer $M(n)$ containing only 1's and 2's in its decimal representation, and $M(n)$ is divisible by 2^n.

5. Find a closed form expression for
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{3n}}{3n}$$

6. Let P_1, $P_2, \ldots$ be an infinite set of points on the x-axis with integer coordinates, and let Q be an arbitrary point in the plane **not** on the x-axis. Prove that infinitely many of the distances $|P_n Q|$ are not integers.