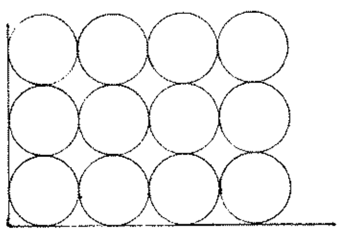
THE TWENTY-FIFTH HERZOG PRIZE EXAMINATION

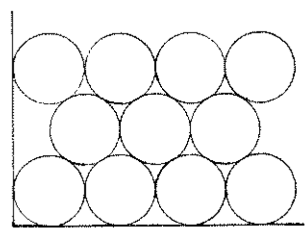
November 8, 1997

1. Prove or disprove: The next-to-last digit of a positive integral power of 3 is always an even number.

2. Prove: There are no integers a, b, c, d such that $f(x) = ax^3 + bx^2 + cx + d$ is equal to 1 when $x = 19$ and equal to 2 when $x = 97$.



Method A



Method B

3. Pictured above are two ways to pack cans of soup in a shallow box. Obviously, method A enables the grocer to pack n^2 cans of unit diameter into an $n \times n$ box. Find the smallest number n for which method B enables the grocer to pack more than n^2 cans into an $n \times n$ box.

4. Let f be a differentiable function such that $f(x) + f'(x) \to 0$ as $x \to \infty$. Prove that $f(x) \to 0$ as well.

5. Find all pairs x, y of positive integers satisfying $x^3 - y^3 = xy + 61$.

6. Let x and y be a pair of positive real numbers satisfying $x^3 + y^3 = x - y$.

a.) Prove that infinitely many such pairs of numbers exist.

b.) Prove that for any such pair, $x^2 + y^2 < 1$.