THE TWENTY-EIGHTH HERZOG PRIZE EXAMINATION

November 4, 2000

1. The recursion formula $f(n+2) = f(n+1) + f(n)$ defines the Fibonacci sequence in case we start with the conditions $f(0) = f(1) = 1$. Of course, if we start with other integral values for $f(0)$ and $f(1)$, different sequences will result. Some of these sequences will contain a multiple of 5 (for example 1,1,2,3,5,... or 8,13,21,34,55,...). However, the starting values $f(0) = 1, f(1) = 3$ result in the sequence 1,3,4,7,11,18,29,..., and it's not immediately clear whether a multiple of 5 will ever make an appearance.
   Determine how many terms should be calculated in order to ascertain whether or not such a sequence will contain a multiple of 5 for *arbitrary* starting values.

2. Determine the locus of all points $P(x,y)$ that satisfy

$$\sqrt{(x-4)^2 + y^2} - \sqrt{x^2 + (y+3)^2} = 7.$$

3. Can a plane circle pass through *exactly* three points with rational coordinates?

4. Let p and q be twin primes greater than 3. Show that $p^2 + q^2 - 2$ is divisible by 72.

5. Each number in the triple $\{112,518,322\}$ is divisible by 14. Furthermore, the determinant

$$\begin{vmatrix} 1 & 1 & 2 \\ 5 & 1 & 8 \\ 3 & 2 & 2 \end{vmatrix} = 14.$$

   Prove the general result: Any common factor of n arbitrary n-digit numbers is also a factor of the analogously formed $n \times n$ determinant.

6. Let $c>1$, and define $F(t) = \int_1^c \frac{dx}{(x^2+t)}$. Prove that $F(t)$ is continuous at $t = 0$.