Herzog Competition 2013

The grading will be harsh, because that is how the Putnam exam is graded. So make sure you justify as much as you can every step! Have fun and may the algorithm be with you!

(1) (OME) Find all integer solutions (x, y) to the equation

$$y^k = x^2 + x$$

where $k > 1$ is a given integer. Recall integers are numbers such as $0, +1, -1, +2, -2, \ldots$.

(2) (SMO) Let $p(x) = x^3 + 2x^2 + 3x + 4$. Find a polynomial of degree 3 whose roots are, precisely, the squares of the roots of p.

(3) (OME) Consider a square of sidelength 10 in the plane, and we move it in such a way that the two vertices of one of its sides always touch the coordinate axes, one of the vertices touches the x-axis, and the other vertex touches the y-axis. Explain, with proof, what is the geometric region that, during this movement of the square, is described by:

(a) the midpoint of the side of the square that is touching the coordinate axes.

(b) the center of the square.

(4) (MMO) In a dance floor there are 15 men and 15 women standing in two parallel rows (all men in one row, all women in another row) in such a way that they make 15 couples. For each couple, the difference in height between the man and the woman is never more than 5 inches. Prove that, if we rearrange the men and the women within each row in increasing order of height, then it will still be true that the height difference between the members of each new couple thus formed will not exceed 5 inches.

(5) (AMO) Let $n \geq 3$ be a natural number. Find out, with proof, which number is bigger, $\ln^2(n)$ or $\ln(n-1)\ln(n+1)$. Recall the natural numbers are $1, 2, 3, 4, ..,$ and $\ln(x)$ is the natural logarithm of x.

(6) (Putnam 1947) A real valued continuous function satisfies for all real x and y the functional equation

$$f(\sqrt{x^2 + y^2}) = f(x)f(y).$$

Prove that

$$f(x) = [f(1)]^{x^2}.$$