Herzog Competition 2014

The grading will be harsh, because that is how the Putnam exam is graded. So make sure you justify as much as you can every step! Have fun and may the algorithm be with you!

(1) In Spain, the plate numbers in cars have 4 digits from 0 to 9, both included. Little Inés, who is learning how to multiply, likes to take the first two digits and multiply them by the last two digits, and with the resulting number, if it has 3 or four digits, she does the same (i.e. she considers a number of 3 digits as one of 4 digits, adding a zero in front of it), over and over. If at any point the resulting number has only two digits, then she only multiplies the first digit by the second digit. If at any point the resulting number has 5 digits, she adds a zero in front of it so that it has 6 digits, and then with any number of 6 digits, she multiplies the first 3 digits by the last 3 digits, and so on. I.e. If a number has an odd number of digits, she adds a zero in front of it to make it an even number of digits, and for any number with an even number of digits, she multiplies the first half of the digits by the second half of them.

E.g. she starts with a plate number 4532, so she does 45x32=1440, so now she does 14x40 = 560 = 0560, so now she does 05x60 = 300, and now 3x00 = 0.

She observes that no matter which 4 digit plate number she starts with, by doing this process, she always seems to finish with a one digit number, after finitely many steps, instead of going to infinity or "wandering around" forever without ever getting a one-digit number. But she doesn't know whether this is true always. Can you prove it?

(2) (MMO) An infinite sequence of numbers x_n, where n runs over the positive integers, is defined by the condition

$$x_{n+1} = 1 - |1 - 2x_n| \ ,$$

where $0 \leq x_1 \leq 1$. Prove that the sequence is eventually periodic if and only if x_1 is rational.

(3) (MMO) Each of the 2014 members of the Parliament of a conflictive country has slapped exactly one colleague in the face. Prove that it is possible to draw from this Parliament a 672-member committee none of whom have slapped one another (i.e. for any two members of such committee, say A and B, A has not slapped B and B has not slapped A).

(4) (OMM) Let D be a point on the side BC of the triangle ABC. Inscribe one circle in each of the triangles ABD and ACD, and draw a common outer tangent to the circles (different from BC). Let its intersection with AD be called K. Prove that the length of AK does not depend on the position of D on BC.

(5) (MMO) Prove that if a, b, and c are integers such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ and $\frac{a}{c} + \frac{c}{b} + \frac{b}{a}$ are also integers, then $|a| = |b| = |c|$.

(6) (Putnam 1953) Let w be an irrational number with $0 < w < 1$. Prove that w has a unique convergent expansion of the form

$$w = \frac{1}{p_0} - \frac{1}{p_0 p_1} + \frac{1}{p_0 p_1 p_2} - \frac{1}{p_0 p_1 p_2 p_3} + \dots,$$

where $p_0, p_1, p_2, \dots$ are integers satisfying $1 \leq p_0 < p_1 < p_2 < \dots$.