THE SIXTH ANNUAL HERZOG PRIZE EXAMINATION

November 4, 1978

Problem 1: (MAA E 1359) Given 8 positive integers $a_1 < a_2 < \cdots < a_8 \leq 16$. Prove that there exists a k such that $a_i - a_j = k$ has at least 3 solutions with $i > j$.

Problem 2: (L.M. Kelly) Let $P(x)$ denote a polynomial with integral coefficients. Prove that it is impossible for $P(1000) = 1000$, $P(2000) = 2000$, and $P(3000) = 1000$.

Problem 3: (Vera T. Sös) Let $0 < \alpha < 1$ be irrational. Put the n fractional parts $\langle \alpha \rangle$, $\langle 2\alpha \rangle$, $\langle 3\alpha \rangle, \ldots, \langle n\alpha \rangle$ in increasing order. Show there is at most 3 distinct distances between successive terms.

Problem 4: If P and Q lie in the interior of a regular tetrahedron, show that the angle PAQ is less than 60° where A is any vertex of the tetrahedron.

Problem 5: (E. Nordhaus) Let S be a set of distinct positive integers such that

1) no integer of S exceeds 100
2) every three integers selected from S can be the sides of a (non-degenerate) triangle.

What is the maximum number of integers S can contain?

Problem 6: (MAA E 1381) Show that $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{2n}}{(2n)!} = 0$ has no real roots.