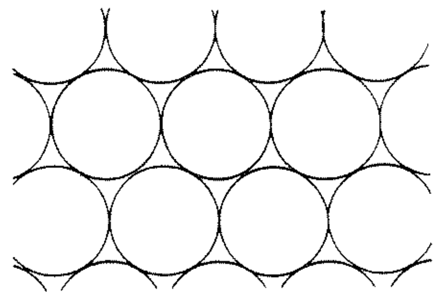
THE TWENTIETH HERZOG PRIZE EXAMINATION

November 14, 1992



1. (See the figure.) The plane is closely packed with circles of unit radius, each circle touching six others, and a point is chosen at random. Find the probability that the point lies in none of the circles.

2. Find all possible p, q and r for which

$$\frac{\log p}{\log q} = r,$$

where p and q are prime and r is rational.

3. Find a closed form expression for the product

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \cdots (1 - \frac{1}{n^2})$$

and prove the correctness of your answer.

4. Three circles of equal radius intersect in a common point. Show that the remaining three intersection points lie on a circle of the same radius.

5. A Pythagorean triple is a set of three positive integers (a, b, c) for which $a^2 + b^2 = c^2$. For example $(3, 4, 5)$ and $(5, 12, 13)$ are Pythagorean triples. Prove that in every Pythagorean triple, at least one of the numbers is a multiple of 5.

6. Prove or disprove: The graph of the equation

$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 1$$

is a bounded subset of the plane.