Herzog Competition
2003

1. A and B are square matrices such that A^{2003} is the zero matrix and $A + B = AB$. Prove that $\det \mathrm{B} = 0$.

2. Let $f : [0,1] \to [0,1]$ be a continuous nondecreasing function. Prove that

$$\int_0^1 f(x)dx \leq \int_0^1 f(f(x))dx + \frac{1}{2}.$$

3. The squares of an 8×8 chessboard are labeled with integers from 1 to 32. There are two squares with each number. Show that one can always find 32 squares with different numbers so that there is at least one of the chosen squares in each row and column.

4. There are two piles of stones with 2003 stones in each. Two players play the following game. Each player can remove one of the piles and split the remaining pile into two nonempty piles. The player who cannot make a move loses. Which player has a winning strategy?

5. a and b are positive integers. Prove that among the numbers $(a + \frac{1}{2})^j + (b + \frac{1}{2})^j$, $j = 1, 2, 3, \cdots$, there are only finitely many integers.