Herzog Competition
2004

1. Let n be a positive integer. Prove that the sum of the digits of 1981^n is at least 19.

2. Let A be a square matrix with real entries such that $A^3 = A + I$. Show that $\det A > 0$.

3. You have a chess board of size 10×10. Prove that there are no more than 2^{50} ways to tile the board with dominoes (i.e., rectangles of size 2×2).

4. Let f be a real continuous function on $[-1, 1]$ such that $|f(t)| \leq 1$ for $t \in [-1, 1]$, $\int_{-1}^{1} f(t)\,dt = 1$, and $\int_{-1}^{1} f^2(t)\,dt = 1$. Show that

(a) $\int_{-1}^{1} f^3(t)\,dt \geq 0$;

(b) $\int_{-1}^{1} f^3(t)\,dt \geq 1/3$.

5. A part of a square of size 1×1 is painted in red. It is known that no two red points of the square are at the distance ε (where $\varepsilon > 0$). Prove that

(a) the area of the red part of the square is at most $\frac{1}{3}(1 + \varepsilon)^2$.

(b) the area of the red part of the square is at most $\frac{2}{7}\left(1 + \sqrt{3}\varepsilon\right)^2$.