Herzog Competition 2012

The grading will be harsh, because that is how the Putnam exam is graded. So make sure you justify as much as you can every step! Have fun and may the algorithm be with you!

(1) (OME) With many equal coins, build a triangle in the following way: place a coin, below it, place 2 coins touching it and themselves, below those 2 coins, place 3 coins touching the previous 2, and so on until you get a triangle filled with coins, whose sides consist of n coins. Compute, in terms of n, the total number of tangencies (touching points) among the coins in the triangle.

(2) (OME) Let a, b, c be three positive real numbers whose product is 1. Assume that the sum of these three numbers is greater than the sum of their reciprocals. Prove that exactly one of the three numbers is larger than 1.

(3) (A-PMO) Show that for any positive integers a and b, the product $(36a + b)(a + 36b)$ cannot be a power of 2.

(4) (H. Hu) The three complex numbers a, b, and c are the zeros of a polynomial $P(z) = z^3 + qz + r$, where q and r are complex numbers, and $|a|^2 + |b|^2 + |c|^2 = 250$. The points corresponding to a, b, and c in the complex plane are the vertices of a right triangle with hypotenuse of length h. Find h^2. Recall that for a complex number $z = x + iy$, with x and y real, $|z|^2 = x^2 + y^2$.

(5) (Y. Wang) Prove the series $\sum_{n=2}^{\infty} (\log n)^{-\log \log n}$ diverges.

(6) (Putnam 1996) Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$