HERZOG MATH COMPETITION

November 14, 2015

Note: The grading will be similar to the grading of Putnam. This means you must explain your answers/steps as much as possible. An answer without any explanation will receive no credit.

1. Let $A = (a_{ij})$ be an $n \times n$ matrix such that $\sum_{j=1}^{n} |a_{ij}| < 1$ for each i. Prove that $\mathrm{I}_n - A$ is invertible. Here I_n is the $n \times n$ identity matrix.

2. Prove that for any non-negative integer n, the number
$$5^{5^{n+1}} + 5^{5^n} + 1$$
is not prime.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that the range of f is all of $\mathbb{R}$.

4. Consider a circle of diameter AB and center O, and the tangent t at B. A variable tangent to the circle with contact point M intersects t at P. Find the locus of the point Q where the line OM intersects the parallel through P to the line AB.

5. A coin is tossed n times. What is the probability that two heads will turn up in succession somewhere in the sequence?

6. Show that the number 2002^{2002} can be written as the sum of four perfect cubes, but not as the sum of three perfect cubes.