On Isometric Embeddings into Anti-de Sitter Spacetimes

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We show that any metric on \( S^2 \) with Gauss curvature \( K \geq -\kappa \) admits a \( C^{1,1} \)-isometric embedding into the hyperbolic space with sectional curvature \( -\kappa \). We also give a sufficient condition for a metric on \( S^2 \) to be isometrically embedded into anti-de Sitter spacetime with the prescribed cosmoligical time function.

1 Introduction

Weyl’s isometric embedding theorem states that the following theorem.

**Theorem 1.1.** Let \( \bar{\sigma} \) be a smooth metric on \( S^2 \) with positive Gauss curvature. Then there exists a smooth isometric embedding \( i:(S^2, \bar{\sigma}) \to \mathbb{R}^3 \), which is unique up to congruence.

Weyl's theorem was proved independently by Nirenberg [15] and Pogorelov [16] using different approaches. Nirenberg used the continuity method that is more familiar to geometric analysts. The Pogorelov approach consists of two steps. First, Pogorelov exhibited a generalized solution as the limit of polyhedra and then Pogorelov proved the regularity of this generalized solution.

Later, Pogorelov generalized the Weyl theorem to hyperbolic space \( \mathbb{H}^3_{-\kappa} \) with sectional curvature \( -\kappa \):
Theorem 1.2 ([17, Theorem 2]). Let \( \bar{\sigma} \) be a smooth metric on \( S^2 \) with Gauss curvature \( K > -\kappa \). Then there exists a smooth isometric embedding \( i : (S^2, \bar{\sigma}) \to \mathbb{H}^3_{-\kappa} \). \( \square \)

Another question concerning Weyl’s theorem is what happens if we assume the Gauss curvature is merely nonnegative instead of positive? Guan and Li [6] and Hong and Zuily [12] independently proved the following theorem (see also [13] for prior results).

Theorem 1.3. Let \( \bar{\sigma} \) be a \( C^4 \) Riemannian metric on \( S^2 \) with Gauss curvature \( K \geq 0 \). Then there exists a \( C^{1,1} \) isometric embedding \( i : (S^2, \bar{\sigma}) \to \mathbb{R}^3 \). \( \square \)

In this paper, we follow Nirenberg’s approach to solve the isometric embedding problem into hyperbolic space by the continuity method. Our main result is the following theorem.

Theorem A. Let \( \bar{\sigma} \) be a smooth metric on \( S^2 \) with Gauss curvature \( K \geq -\kappa \). Then there exists a \( C^{1,1} \) isometric embedding \( i : (S^2, \bar{\sigma}) \to \mathbb{H}^3_{-\kappa} \). \( \square \)

Given an embedding \( i : S^2 \to \mathbb{H}^3_{-\kappa} \) and a trivialization \( \varphi : \mathbb{H}^3_{-\kappa} \to \mathbb{R}^3 \), let \( r = \varphi \circ i \) be the “position vector”. After choosing local coordinates \( \{u^a\} \), \( a = 1, 2 \) on \( S^2 \), the isometric embedding equation can be written as the nonlinear first-order partial differential system

\[
g_{ij}(r(x)) \frac{\partial r^i}{\partial u^a}(x) \frac{\partial r^j}{\partial u^b} = \bar{\sigma}_{ab}(x). \tag{1}\]

Unlike the case of Euclidean space, the form of the equation depends on the choice of the trivialization. For most of this paper, we choose to use the static coordinate chart of \( \mathbb{H}^3_{-\kappa} \) in which the manifold is identified with \( \mathbb{R}^3 \) and the metric has the form

\[
g = \frac{1}{f^2} \, dr^2 + r^2 g_{S^2},
\]

where \( f = \sqrt{1 + \kappa r^2} \) is called the static potential and \( g_{S^2} \) denotes the standard metric on \( S^2 \).

We now outline the proof of Theorem A. First of all, the normalized Ricci flow [5, 9] provides an one-parameter family of smooth metrics \( \sigma_t, t \in [0, \infty) \) on \( S^2 \) such that \( \sigma_0 = \bar{\sigma} \) and \( \sigma_t \) converges to a metric \( \sigma_\infty \) with constant Gauss curvature. Moreover, the Gauss curvature of \( \sigma(t) \) is greater than \( -\kappa \) for \( t > 0 \).

Let \( I \subset [0, \infty) \) be the set of parameters such that \( \sigma_t \) can be isometrically embedded into \( \mathbb{H}^3_{-\kappa} \) as a closed convex \( C^{1,1} \) surface. The goal is to show that \( I \) is nonempty, open, and closed. As a result, \( I = [0, \infty) \).
We start with the openness part which states that if a metric $\sigma$ can be isometrically embedded into $\mathbb{H}_3^{-\kappa}$, so can any small perturbation of $\sigma$. The first step is to understand the infinitesimal deformation. We show that all infinitesimal deformations come from the isometry of hyperbolic space. The next step is to solve the linearized equation. Remarkably, Nirenberg reduced the system of equations to a single scalar equation and used the Hilbert theory to solve it. We modify Nirenberg’s argument so that it fits the geometry of hyperbolic space. At last, we utilize the contraction mapping principle to solve the nonlinear equation.

Nonemptiness follows from openness. Since the limit metric $\sigma_\infty$ can be isometrically embedded into $\mathbb{H}_3^{-\kappa}$ (with the same image as a round sphere), $(T_0, \infty) \subset I$ for some large $T_0$.

The closedness part boils down to an a priori estimate of the mean curvature $H$ of convex surfaces in $\mathbb{H}_3^{-\kappa}$. Weyl’s original approach is to apply the maximum principle to the mean curvature of the surface, which fails when the metric has negative Gauss curvature somewhere. We manage to find a test function to overcome the difficulty. We show that the following theorem.

**Theorem B.** Let $\Sigma$ be a closed convex surface in $\mathbb{H}_3^{-\kappa}$, normalized so that $\Sigma$ is centered at the origin. Then

$$\max \Sigma H \leq C,$$

for some constant $C$ depending only on $\|f\|_{C^0(\Sigma)}$, and $\|K\|_{C^2(\Sigma)}$. Here, $f = \sqrt{1 + \kappa r^2}$ is the static potential and $K$ denotes the Gauss curvature of $\Sigma$. \hfill \square

During the preparation of this paper, we learned that Guan and Lu [7] independently obtained the same estimate as in Theorem B. Note that for convex surfaces in hyperbolic space with sectional curvature $-1$, Chang and Xiao [4] proved an a priori bound on the mean curvature with an argument based on Pogorelov’s estimate using the additional assumption that the set $\{K = -1\}$ consists of only finitely many points. Our new argument does not require this finiteness condition.

With Nirenberg’s estimates for uniformly elliptic equations in two dimensions, we show that if $\tilde{\sigma}$ has $K > -\kappa$, the isometric embedding constructed by the continuity method is actually smooth. Hence we recover Pogorelov’s result.

The Weyl theorem plays a prominent role in the study of quasilocal mass in general relativity. Let $\Sigma$ be a 2-surface in an initial data set. Suppose the induced metric $\sigma$ has positive Gauss curvature. By Weyl’s theorem, there exists a unique (up to
congruence) isometric embedding \( i : (\Sigma, \sigma) \to \mathbb{R}^3 \). Let \( H_0 \) and \( H \) be the mean curvature of \( i(\Sigma) \subset \mathbb{R}^3 \) and \( \Sigma \) in the initial data set, respectively. The Brown–York mass \([2, 3]\) is defined as

\[
m_{BY}(\Sigma) = \frac{1}{8\pi} \left( \int_{\Sigma} H_0 \, d\mu - \int_{\Sigma} H \, d\mu \right).
\]

In [20], Wang and Yau proposed a new quasilocal mass for space-like 2-surfaces in spacetime. The definition requires the existence of an isometric embedding of the surface into Minkowski spacetime. It is also interesting to consider other backgrounds. Theorem C provides a necessary condition for the existence of isometric embeddings into the cosmological chart of anti-de Sitter spacetime with prescribed cosmological time function.

**Theorem C.** Given a smooth metric \( \sigma \) and a smooth function \( s \) on \( S^2 \). Let \( \nabla \) and \( \Delta \) denote the gradient and Laplace operator with respect to \( \sigma \), respectively. Suppose the Gauss curvature \( K \) and the function \( s \) satisfy

\[
K + \frac{S'}{S} \Delta s - \left( \frac{S''}{S} - \frac{S^2}{S^2} \right) |\nabla s|^2 + (1 + |\nabla s|^2)^{-1} \left( \frac{\det(\nabla^2 s)}{\det \sigma} - \frac{S'}{S} \nabla^a s \nabla^b s \nabla_a \nabla_b s \right) > -\kappa
\]

where \( S(t) = \cos(\sqrt{\kappa} t) \) is the scale factor of anti-de Sitter spacetime in its cosmological chart. Then there exists a unique space-like isometric embedding \( i : (S^2, \sigma) \to AdS \) with prescribed cosmological time function \( s \). \( \square \)

The rest of this paper is organized as follows. In Section 2, we describe the Killing and conformal Killing vector fields on hyperbolic space and list the necessary formulae. In Section 3, we study the infinitesimal rigidity of convex surfaces in \( \mathbb{H}^3_{-\kappa} \). Section 4 is the most important of the paper. In this section, we construct a path of metrics and prove openness on this path. Then, we establish the a priori estimate that proves Theorem B. Together these results provide a proof of Theorem A. Finally, in the last section we discuss isometric embeddings into the anti-de Sitter spacetime and the proof of Theorem C.

## 2 The Killing Vectors on Hyperbolic Space

In the hyperboloid model, \( \mathbb{H}^n_{-\kappa} \) is identified as a hypersurface in \( \mathbb{R}^{n,1} \) given by \(-(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = -\frac{1}{\kappa}\). The projection of the hyperboloid on to the hyperplane \( \{x_0 = 0\} \) provides the static coordinates \( (\mathbb{R}^n, g) \) of \( \mathbb{H}^n_{-\kappa} \). In polar coordinates, the induced metric
is expressed as
\[ g = \frac{1}{1 + \kappa r^2} \, dr^2 + r^2 g_{S^{n-1}}. \]

The hyperbolic space admits a conformal Killing vector field
\[ X = r\sqrt{1 + \kappa r^2} \frac{\partial}{\partial r} = rf \frac{\partial}{\partial r}, \quad (3) \]
where as earlier the function \( f = \sqrt{1 + \kappa r^2} \) is the static potential. In the hyperboloid model, the static potential can be rewritten as \( f = \sqrt{\kappa} x^0 \) and we have that
\[ X = \sqrt{\kappa} \left( \left( (x^0)^2 - \frac{1}{\kappa} \right) \frac{\partial}{\partial x^0} + x^0 \sum_{i=1}^{3} x^i \frac{\partial}{\partial x^i} \right). \]

Let \( D \) denote the covariant derivative on the hyperbolic space.

**Proposition 2.1.** The gradient of the static potential \( f = \sqrt{1 + \kappa r^2} \) and the covariant derivative of the conformal Killing vector field \( X = rf \frac{\partial}{\partial r} \) on \( \mathbb{H}^n_{-\kappa} \) are the following:
\[ Df = \kappa X; \quad (4) \]
\[ D_\xi X = f \xi. \quad (5) \]

**Proof.** We calculate in polar coordinates. Let \( \theta^i, \ i = 1, \ldots, n-1 \) be any local coordinates on \( S^{n-1} \). We write \( \xi = \xi^r \frac{\partial}{\partial r} + \xi^i \frac{\partial}{\partial \theta^i} \).

\[ Df = g^{rr} \partial_r f \frac{\partial}{\partial r} = \kappa rf \frac{\partial}{\partial r}, \]
\[ D_\xi X = D_\xi \left( rf \frac{\partial}{\partial r} \right) = \xi^r f \frac{\partial}{\partial r} + \frac{\kappa r^2}{f} \xi^r \frac{\partial}{\partial r} + rf \left( \xi^r D_\xi \frac{\partial}{\partial r} + \xi^i D_\xi \frac{\partial}{\partial \theta^i} \right) = f \left( \xi^r \frac{\partial}{\partial r} + \xi^i \frac{\partial}{\partial \theta^i} \right). \]

On the hyperboloid, besides the obvious rotational symmetry, there are translational symmetries coming from the isometric action of the off-diagonal part of \( O(1, n) \).
We take a curve tangent to the hyperboloid:
\[
\begin{pmatrix}
1 & t & 0 & 0 \\
t & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
& & & \ddots \\
0 & & & 1
\end{pmatrix}
\begin{pmatrix}
x^0 \\
x^1 \\
\vdots \\
x^n
\end{pmatrix},
\]
and project the curve on to \( \mathbb{R}^n \). The tangent vector of this curve gives a Killing vector field \( \frac{f}{\sqrt{\kappa}} \frac{\partial}{\partial x^1} \) at \((x^1, x^2, \ldots, x^n)\). Hence any translational Killing vector field in the static coordinates is of the form \( f Z_0 \) for some constant vector field \( Z_0 = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \).

On Euclidean space, the constant vector field is characterized by \( d Z = 0 \). On hyperbolic space, we introduce a twisted covariant derivative to get a similar characterization.

**Definition 2.2.** The twisted covariant derivative on \( \mathbb{H}^n_{-\kappa} \) is defined as
\[
\tilde{D}_\xi Z = D_\xi Z + \frac{\kappa}{f} (\langle \xi, Z \rangle) X,
\]
where \( (\cdot, \cdot) = g(\cdot, \cdot) \).

**Proposition 2.3.** The constant vector field in hyperbolic space is characterized by \( \tilde{D} Z = 0 \). That is, \( \tilde{D} Z = 0 \) if and only if \( Z = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \) in the static coordinates.

**Proof.** We work in the hyperboloid model. From (6), every constant vector is a linear combination of \( \{ \frac{x^i}{x^0} \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^0} \}_{i=1}^n \). Moreover, the tangent space is spanned by translational Killing vectors \( \{ x^i \frac{\partial}{\partial x^i} + x^0 \frac{\partial}{\partial x^0} \}_{i=1}^n \). Therefore, it suffices to check
\[
\tilde{D}_{x^i} \frac{\partial}{\partial x^0} + x^0 \frac{\partial}{\partial x^i}
\]
and
\[
\frac{x^i}{x^0} \frac{\partial}{\partial x^i} + x^0 \frac{\partial}{\partial x^0} + \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}
\]

\[
= \left( -\frac{x^i x^i}{(x^0)^2} + \delta^{ij} \right) \frac{\partial}{\partial x^j}
\]

\[
= \left( -\frac{x^i x^i}{(x^0)^2} + \delta^{ij} \right) \frac{\partial}{\partial x^0} - \kappa \left( \frac{x^i x^i}{x^0} + x_0 \delta^{ij} \right) \left( x^0 \frac{\partial}{\partial x^0} + \sum_{i=1}^n x^i \frac{\partial}{\partial x^i} \right)
\]

\[
+ \kappa \left( \frac{x^i x^i}{(x^0)^2} + \delta^{ij} \right) \left( (x^0)^2 - \frac{1}{\kappa} \right) \frac{\partial}{\partial x^0} + x^0 \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}
\]

\[
= 0,
\]
where the superscript $T$ denotes the tangent component of a vector. On the other hand, given a vector field $Z = \sum_{i=1}^{n} a_i(x) \left( \frac{\partial}{\partial x^0} + \sum_{j=1}^{n-1} \frac{\partial}{\partial x^j} \right)$,

$$\tilde{D}_\xi Z = \sum_{i=1}^{n} \xi(\partial_{\xi} a_i) \left( \frac{\partial}{\partial x^0} + \sum_{j=1}^{n-1} \frac{\partial}{\partial x^j} \right).$$

Therefore $\tilde{D}Z = 0$ if and only if $Z$ is a constant vector field.

3 The Infinitesimal Rigidity

In this section, we prove the infinitesimal rigidity of convex surfaces, which will be used in the openness part of Theorem A.

We start with some general discussion on infinitesimal deformations. Let $r : (N, \sigma) \rightarrow (M, g)$ be an isometric embedding of a hypersurface and $\Sigma$ be the image of the embedding. Let $E = r^{-1}(TM)$ be the pull-back of the tangent bundle over $N$. We abuse notation to denote the pull-back connection by $D$. Denote the differential of $r$ by $Dr$ and view it as a section $Dr \in \Gamma(N, E \otimes T^*N)$. We use the Einstein summation convention of summing repeated indices. The indices $a, b, c = 1, 2, \ldots, n-1$ and $i, j = 1, 2, \ldots, n$. Let $\{n^i\}$ and $\{x^i\}$ be local coordinates on $N$ and $M$, respectively. Let $\{e_a\}$ denote a local orthonormal frame such that $\{e_a\}$ are tangent to $N$ and $v$ is the unit outward normal. Let $\{\omega^a\}$ be the dual 1-form of $\{e_a\}$.

**Definition 3.1.** For $v_1 \otimes \omega_1, v_2 \otimes \omega_2 \in \Gamma(N, E \otimes T^*N)$, we define

$$(v_1 \otimes \omega_1) \odot (v_2 \otimes \omega_2) := \frac{1}{2}g(v_1, v_2)(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)$$

and extend it linearly.

Suppose there is a family of isometric embeddings $r_t$ with $r_0(x) = r(x)$. We say that $r_t$ yields a first-order isometric deformation of $r$ if

$$\frac{d}{dt}(Dr_t \odot Dr_t) \bigg|_{t=0} = 0.$$

Set $\tau = \frac{dr_t}{dt} \big|_{t=0} \in \Gamma(N, E)$. We have the following equivalent equation.

**Lemma 3.2.** The infinitesimal deformation equation is given by

$$Dr \odot D\tau = 0,$$  \hspace{1cm} (7)

where $\odot$ is the symmetric product of one-forms. We call $\tau$ an infinitesimal deformation (or an isometric deformation).
Proof. We use \( \{u^a\} \) and \( \{x^i\} \) to denote local coordinates on \( N \) and \( M \), respectively. Then
\[
Dr = \frac{\partial r^i}{\partial u^a} \frac{\partial}{\partial x^i} \otimes du^a, \quad \tau = \left. \frac{\partial r^i}{\partial t} \right|_{t=0} \frac{\partial}{\partial x^i},
\]
and
\[
\sigma_{ab} du^a \otimes du^b = Dr_t \circ Dr_t(p)
= \frac{\partial r_i}{\partial u^a}(p) \frac{\partial r_j}{\partial u^b} g_{ij}(r_t(p)) du^a \otimes du^b.
\]
Differentiating with respect to \( t \) and evaluating at \( t = 0 \), we obtain
\[
0 = \left. \left( \frac{\partial^2 r^i}{\partial u^a \partial t} \frac{\partial r^j}{\partial u^b} g_{ij} + \frac{\partial r^i}{\partial u^a} \frac{\partial^2 r^j}{\partial u^b \partial t} g_{ij} + \frac{\partial r^i}{\partial u^a} \frac{\partial r^j}{\partial u^b} \frac{\partial g_{ij}}{\partial x^k} \frac{\partial}{\partial t} \right) \right|_{t=0} du^a \otimes du^b
= \left( \frac{\partial \tau^i}{\partial u^a} \frac{\partial r^j}{\partial u^b} g_{ij} + \frac{\partial r^i}{\partial u^a} \frac{\partial \tau^j}{\partial u^b} g_{ij} + \frac{\partial r^i}{\partial u^a} \frac{\partial r^j}{\partial u^b} \frac{\partial g_{ij}}{\partial x^k} \tau^k \right) du^a \otimes du^b.
\]
On the other hand,
\[
D\tau = \left( \frac{\partial \tau^i}{\partial u^b} + \frac{\partial r^m}{\partial u^b} \Gamma^j_{mk} \right) \frac{\partial}{\partial x^i} \otimes du^b.
\]
We then have
\[
Dr \circ D\tau = \frac{1}{2} \left( \frac{\partial r^i}{\partial u^a} \left( \frac{\partial r^j}{\partial u^b} \frac{\partial \tau^k}{\partial u^m} + \frac{\partial r^m}{\partial u^b} \Gamma^j_{mk} \right) g_{ij} + (a, b \text{ symmetric}) \right) du^a \otimes du^b
= \frac{1}{2} \left( \frac{\partial \tau^i}{\partial u^a} \frac{\partial r^j}{\partial u^b} g_{ij} + \frac{\partial r^i}{\partial u^a} \frac{\partial \tau^j}{\partial u^b} g_{ij} + \frac{\partial r^i}{\partial u^a} \frac{\partial r^j}{\partial u^b} (\Gamma^j_{mk} g_{ij} + \Gamma^j_{ik} g_{mj}) \tau^k \right) du^a \otimes du^b
= \frac{1}{2} \left( \frac{\partial \tau^i}{\partial u^a} \frac{\partial r^j}{\partial u^b} g_{ij} + \frac{\partial r^i}{\partial u^a} \frac{\partial \tau^j}{\partial u^b} g_{ij} + \frac{\partial r^i}{\partial u^a} \frac{\partial r^j}{\partial u^b} \frac{\partial g_{ij}}{\partial x^k} \tau^k \right) du^a \otimes du^b
= 0,
\]
where in the second to last equality \( i, m \) are symmetric as \( a, b \) are symmetrized. □

Definition 3.3. A solution \( \tau \in \Gamma(N, E) \) of (7) is called an infinitesimal deformation. An infinitesimal deformation is trivial if it is the restriction of some Killing vector field of \( M \) on the hypersurface. An isometric embedding is called infinitesimally rigid if it only has trivial infinitesimal deformations. □

We now focus on the isometric embedding into 3D hyperbolic space. Take \( N = S^2 \) and \( M = \mathbb{H}^3_\kappa \). From (5), the “position vector” \( r \) can be replaced by the conformal Killing
vector $X$ up to the static potential, which is nonzero. We henceforth write the infinitesimal deformation equation as

$$DX \odot D\tau = 0. \tag{8}$$

The main result in this section is the following theorem.

**Theorem 3.4.** Consider an isometric embedding $r : (S^2, \sigma) \to (\mathbb{H}^3_{-\kappa}, g)$. Suppose that the image $\Sigma$ is a convex hypersurface in $\mathbb{H}^3_{-\kappa}$. Then $r$ is infinitesimally rigid. $\square$

Before proving this theorem, we need to establish some basic results. The fiber of $E$ is a 3D inner product space. We can define the usual cross product $\times$ once we fix an orientation. The identity $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$ will be useful. First we need a definition.

**Definition 3.5.** We define the inner product and cross product for differential forms valued in $E$ by

$$(v_1 \otimes \omega_1, v_2 \otimes \omega_2) = \langle v_1, v_2 \rangle \otimes (\omega_1 \wedge \omega_2),$$

$$(v_1 \otimes \omega_1) \times (v_2 \otimes \omega_2) := (v_1 \times v_2) \otimes (\omega_1 \wedge \omega_2),$$

and extend it linearly. We also have the pull-back twisted covariant derivative $\tilde{D} : \Gamma(S^2, E) \to \Gamma(S^2, E \otimes T^*S^2)$ defined as

$$\tilde{D}_{\xi} Z := D_{\xi} Z + \frac{\kappa}{f}(r_* (\xi), Z) X$$

for $\xi \in TS^2$ and $Z \in \Gamma(S^2, E)$. $\square$

Next, we need the following lemmas.

**Lemma 3.6.** Suppose $\tau$ is an infinitesimal deformation. Then

$$\tilde{D} \left( \frac{\tau}{f} \right) = Y \times DX$$

for some $Y$, which is called the rotation vector of $\tau$. $\square$
Proof. Choose a local orthonormal frame \( \{ e_1, e_2, \nu \} \). Let \( \omega^1, \omega^2 \) be the dual of \( e_1, e_2 \). By (4), we have

\[
\tilde{\mathcal{D}} \left( \frac{\tau}{f} \right) = \left[ D_{e_a} \left( \frac{\tau}{f} \right) + \frac{\kappa}{f^2} \left( e_a - e_a \right) \right] \omega^a
\]

\[
= \left[ D_{e_a} \frac{\tau}{f} + e_a \left( \frac{1}{f} \right) \tau + \frac{\kappa}{f^2} (e_a, \tau) e_i \right] \omega^a
\]

\[
= \left[ D_{e_a} \frac{\tau}{f} - \frac{\kappa}{f^2} (X, e_a) (\tau, e_i) e_i + \frac{\kappa}{f^2} (\tau, e_a) \langle X, e_i \rangle e_i \right] \omega^a
\]

\[
= [A_{a b} e_b + B_a \nu] \omega^a.
\]

From (8),

\[
D_{e_a} \cdot e_b + D_{e_b} \cdot e_a = 0.
\]

Hence \( A_{a b} \) is anti-symmetric and \( \tilde{\mathcal{D}} (\frac{\tau}{f}) = Y \times DX \) for \( Y = \frac{1}{f} (B_2 e_1 - B_1 e_2 + A_{12} \nu) \).

When we restrict a Killing vector field of hyperbolic space to the surface, we get an infinitesimal deformation \( \tau = Y_0 \times X + f Z_0 \). We compute the rotation vector for such \( \tau \).

Lemma 3.7. For a Killing vector \( \tau = Y_0 \times X + f Z_0 \), where \( Y_0 \) and \( Z_0 \) are constant vectors, its rotation vector is \( \frac{1}{f} (Y_0 + \kappa (Y_0, X) X) \).

Proof. By Proposition 2.3, \( \tilde{\mathcal{D}} Y_0 = 0 \) and \( \tilde{\mathcal{D}} Z_0 = 0 \). Thus,

\[
\tilde{\mathcal{D}} \left( \frac{\tau}{f} \right) = \frac{Y_0}{f} \times DX + \left( (Y_0 \times X) e_a \left( \frac{1}{f} \right) + \frac{\kappa}{f} (e_a, \frac{Y_0 \times X}{f}) \right) X \omega^a
\]

\[
= \frac{Y_0}{f} \times DX + \frac{\kappa}{f^3} ((DX, Y_0 \times X) X - (DX, X) (Y_0 \times X))
\]

\[
= \frac{Y_0}{f} \times DX + \frac{\kappa}{f^3} ((Y_0 \times X) \times X) \times DX
\]

This implies that

\[
Y = \frac{Y_0}{f} + \frac{\kappa}{f^3} (Y_0 \times X) \times X
\]

\[
= \frac{Y_0}{f} + \frac{\kappa}{f^3} ((Y_0, X) X - (X, X) Y_0)
\]
\[ \frac{Y_0}{f} \left( 1 - \frac{\kappa r^2}{f^2} \right) + \frac{\kappa}{f^3} (Y_0, X) X = \frac{1}{f^3} (Y_0 + \kappa (Y_0, X) X). \]

**Lemma 3.8.** A vector \( Z \) can be written as \( Z_0 + \kappa (Z_0, X) X \) for some constant vector \( Z_0 \) if and only if
\[
DZ = \frac{\kappa}{f^2} (Z, X) DX.
\]

**Proof.** The only if part follows by direct computation. For the if part, let \( W = D_\xi Z - \frac{\kappa}{f} (Z, X) \xi = 0 \). We then compute
\[
\tilde{D}_\xi \left( Z - \frac{\kappa}{f^2} (Z, X) X \right) = W - \frac{\kappa}{f^2} (W, X) X = 0.
\]
Hence \( Z = \frac{\kappa}{f^2} (Z, X) X + Z_0 \) for some constant vector \( Z_0 \). The assertion follows by taking inner product with \( X \).

In terms of the cross product, we express the Riemann curvature tensor on hyperbolic space as
\[
R(X, Y)Z = -\kappa (Y, Z) X + \kappa (X, Z) Y = \kappa Z \times (Y \times X).
\]

Let \( \tilde{Y} = f^3 Y \), where \( Y \) is the rotation vector of \( \tau \).

**Lemma 3.9.** \( D\tilde{Y} \) is tangential, that is, \( D\tilde{Y} = C^b_a e_b \omega^a \) for some \( 2 \times 2 \) matrix \( C^b_a \). Moreover, \( 2\kappa (\tilde{Y}, X) = f(C^1_1 + C^2_2) \).

**Proof.** From Lemma 3.6, \( \tilde{D}(\tilde{Y}) = D(\tilde{Y}) + \frac{\kappa}{f} (\tilde{Y}, X) X = Y \times DX \). Choosing an orthonormal frame \( e_1, e_2 \) such that the tangential component \( D_{e_a}^T e_b(p) = 0 \). Multiplying by \( f^3 \) and taking derivative, we have at \( p \),
\[
f^3 D_{e_b} D_{e_a} \frac{\tau}{f} e_b(f^3) D_{e_a} \frac{\tau}{f} + D_{e_b}(\kappa f(\tau, e_a) X) = (D_{e_b} \tilde{Y}) \times D_{e_a} X + \tilde{Y} \times D_{e_b} D_{e_a} X.
\]
Antisymmetrizing $a,b$ and using the curvature identity, we get
\[
f^3 \left( -\kappa \left< e_a, \frac{\tau}{f} \right> e_b + \kappa \left< e_b, \frac{\tau}{f} \right> e_a \right) + e_b(f^3)De_a \left( \frac{\tau}{f} \right) - e_a(f^3)De_b \left( \frac{\tau}{f} \right) + e_b(\kappa f(\tau, e_a))X - e_a(\kappa f(\tau, e_b))X + \kappa f(\langle \tau, e_a \rangle De_a X - \langle \tau, e_b \rangle De_b X) \\
= De_b \bar{Y} \times De_a X - De_a \bar{Y} \times De_b X + \bar{Y} \times (-\kappa \langle e_a, X \rangle e_b + \kappa \langle e_b, X \rangle e_a).
\]

Note the first and last term of the left-hand side cancel. By Lemma 3.6, (5), and (4),
\[
e_b(f^3)Y \times De_a X - e_a(f^3)Y \times De_b X = 2\kappa f^3 \langle \bar{Y}, \nu \rangle X - 2\kappa \langle \bar{Y}, (e_a \times e_b) \rangle X - 2\kappa \langle \bar{Y}, (e_a \times e_b) \rangle X.
\]

Therefore,
\[
D_{e_b} \bar{Y} \times De_a X - De_a \bar{Y} \times De_b X \\
= 2\kappa f^3 \langle \bar{Y}, \nu \rangle X - 2\kappa \langle \bar{Y}, (e_a \times e_b) \rangle X.
\]

Let $a = 1$, $b = 2$. We have
\[
D_{e_2} \bar{Y} \times fe_1 - D_{e_1} \bar{Y} \times fe_2 = 2\kappa \bar{Y} \times (X \times \nu) - 2\kappa \langle \bar{Y}, \nu \rangle X = -2\kappa \langle \bar{Y}, X \rangle \nu.
\]

The lemma follows by comparing the tangential and normal components.
Proof. By straightforward computation, we have

\[ f^2 \psi = -df \cdot \tilde{Y} \times \left( D\tilde{Y} - \frac{\kappa}{f^2} \langle \tilde{Y}, X \rangle DX \right) \]

\[ + fDX \cdot \left( \tilde{Y} - \frac{\kappa}{f^2} \langle \tilde{Y}, X \rangle X \right) \times \left( D\tilde{Y} - \frac{\kappa}{f^2} \langle \tilde{Y}, X \rangle DX \right) \]

\[ + fX \cdot \left( D\tilde{Y} - \frac{\kappa}{f^2} \langle \tilde{Y}, X \rangle DX \right) \times \left( D\tilde{Y} - \frac{\kappa}{f^2} \langle \tilde{Y}, X \rangle DX \right) \]

\[ + fX \cdot \tilde{Y} \times \left( D^2\tilde{Y} - d\left( \frac{\kappa}{f^2} \langle \tilde{Y}, X \rangle \right) \wedge DX - \frac{\kappa}{f^2} \langle \tilde{Y}, X \rangle D^2 X \right) \]

\[ = I + II + III + IV. \]

Here

\[ D^2\tilde{Y} = \frac{1}{2} (D_{e_a} D_{e_b} \tilde{Y} - D_{e_b} D_{e_a} \tilde{Y} - D_{[e_a, e_b]} \tilde{Y}) \omega^a \omega^b \]

\[ = -\kappa (\tilde{Y} \times v) \omega^1 \wedge \omega^2. \]

by the curvature identity.

We claim that II = 0. Indeed, we have

\[ DX \times D\tilde{Y} = f(C_1 + C_2)^v \omega^1 \wedge \omega^2 \]

\[ DX \times DX = 2 f^2 v \omega^1 \wedge \omega^2. \]

By Lemma 3.9, II = 0. Consequently, it remains to show that I + IV = 0

The following identity can be verified directly by expanding each term in an orthonormal frame.

\[ v \times (\alpha \times \beta) = -\langle v, \alpha \rangle \beta - \langle v, \beta \rangle \alpha \]

for any \( v \in \Gamma(S^2, E) \) and \( \alpha, \beta \in \Gamma(S^2, E \otimes T^* S^2) \).

From (4), we have

\[ df = \frac{\kappa}{f} \langle X, DX \rangle \]

\[ d \left( \frac{\kappa}{f^2} \langle \tilde{Y}, X \rangle \right) = \frac{\kappa}{f^2} \left( -\frac{\kappa}{f^2} \langle X, DX \rangle \langle \tilde{Y}, X \rangle + \langle D\tilde{Y}, X \rangle + \langle \tilde{Y}, DX \rangle \right) \]
Hence \( I + IV = \frac{5}{4} X \cdot \bar{Y} \times \Omega \) where

\[
\Omega = \left( -(X, DX) \wedge D \bar{Y} + \frac{\kappa}{f^2} \langle \bar{Y}, X \rangle \langle X, DX \rangle \wedge DX - f^2(\bar{Y} \times v)\omega^1 \wedge \omega^2 \right.
\]
\[
+ \left. \frac{2\kappa}{f^2} \langle \bar{Y}, X \rangle \langle X, DX \rangle \wedge DX - \langle X, D \bar{Y} \rangle \wedge DX - \langle \bar{Y}, DX \rangle \wedge DX + \kappa \langle \bar{Y}, X \rangle (X \times v)\omega^1 \wedge \omega^2 \right)
\]

From (11), (10), and (9), we obtain

\[
\Omega = \left( 2\kappa \langle \bar{Y}, X \rangle (X \times v) - \kappa \langle \bar{Y}, X \rangle (X \times v) - f^2(\bar{Y} \times v) \right.
\]
\[
- \left. 2\kappa \langle \bar{Y}, X \rangle (X \times v) + f^2(\bar{Y} \times v) + \kappa \langle \bar{Y}, X \rangle (X \times v) \right) \omega^1 \wedge \omega^2
\]
\[
= 0 \quad \square
\]

We are ready to prove the rigidity result in Theorem 3.4.

**Proof of Theorem 3.4.** From Lemma 3.9, we can write

\[
D \bar{Y} - \frac{\kappa}{f^2} (\bar{Y}, X) DX = B_b^a e_a \omega^b
\]

with \( B_1^1 + B_2^2 = 0 \). Choosing an orthonormal frame \( e_1, e_2 \) such that the tangential component \( D_{e_a} e_b(p) = 0 \). At \( p \), we have

\[
D_{e_a} \left( D_{e_b} \bar{Y} - \frac{\kappa}{f^2} (\bar{Y}, X) D_{e_b} X \right) = e_a (B_b^c) e_c - B_b^c h_{ac} v,
\]

where \( h_{ab} \) is the second fundamental form of \( \Sigma \). Antisymmetrizing and using the curvature identity, we obtain

\[
- \kappa \bar{Y} \times (e_a \times e_b) + \alpha e_b - \beta e_a + \frac{\kappa^2}{f^2} (\bar{Y}, X) X \times (e_a \times e_b)
\]
\[
= (e_a (B_b^c) - e_b (B_a^c)) e_c + (B_b^c h_{bc} - B_b^c h_{ac}) v.
\]

Here \( \alpha = - f D_{e_a} (\frac{\kappa}{f^2} (\bar{Y}, X)) \) and \( \beta = - f D_{e_b} (\frac{\kappa}{f^2} (\bar{Y}, X)) \). Let \( a = 1, b = 2 \). Comparing the normal component, we have \( B_1^c h_{2c} - B_2^c h_{1c} = 0 \). In matrix form,

\[
\text{Tr} \left( \begin{pmatrix} B_1^1 & B_1^2 \\ B_2^1 & B_2^2 \end{pmatrix} \begin{pmatrix} h_{21} & -h_{11} \\ h_{22} & -h_{12} \end{pmatrix} \right) = 0.
\]

By the Gauss equation, the determinant of the second matrix is \( K + \kappa \). Since \( (\text{Tr}(M))^2 \geq 2 \det(M) \) for any \( 2 \times 2 \) matrix, \( \det(B) \leq 0 \). On the other hand, Lemma 3.10 and Stoke's
theorem imply
\[
0 = \int_{\Sigma} \psi
= \int_{\Sigma} d \left[ \frac{1}{f} X \cdot \left( \tilde{\nabla} - \frac{\kappa}{f^2} \langle \tilde{\nabla}, X \rangle X \right) \times \left( D \tilde{\nabla} - \frac{\kappa}{f^2} \langle \tilde{\nabla}, X \rangle DX \right) \right]
= \int_{\Sigma} \frac{X}{f} \cdot (B^a_{b} e_a \omega^b) \times (B^d_{e} c_d \omega^d)
= \int_{\Sigma} 2 \langle X, \nu \rangle f \det B \omega^1 \wedge \omega^2.
\]
Hence, \( \det(B) \equiv 0 \). We may assume \( h_{11} h_{22} > 0, h_{12} = h_{21} = 0 \). Equation (13) then implies \( B^1_1 B^2_2 \geq 0 \). Since \( B^1_1 + B^2_2 = 0, B \equiv 0 \).

By Lemma 3.8, \( \tilde{\nabla} = Z_0 + \kappa \langle Z_0, X \rangle X \) for some constant vector \( Z_0 \). From the definition of \( \tilde{\nabla} \), \( \tilde{D}(\frac{t}{f}) = \frac{1}{f} (Z_0 + \kappa \langle Z_0, X \rangle X) \times DX \). By Lemma 3.7, \( \tilde{D}(\frac{t-Z_0 \times X}{f}) = 0 \).
Hence \( \tau = Z_0 \times X + f W_0 \) for some constant vector \( W_0 \). This completes the proof of Theorem 3.4.

\[ \square \]

4 Isometric Embeddings into \( \mathbb{H}^3_{-\kappa} \)

The main goal of this section is to prove the following theorem.

**Theorem 4.1 (Theorem A).** Let \( \tilde{\sigma} \) be a smooth metric on \( S^2 \) with Gauss curvature \( K \geq -\kappa \). Then there exists a \( C^{1,1} \) isometric embedding into \( \mathbb{H}^3_{-\kappa} \).

We prove the theorem by the continuity method. It consists of three steps:

1. **Connectedness:** show that there exists a family of smooth metrics \( \sigma_t, t \in [0, \infty) \) such that \( \sigma_0 = \tilde{\sigma} \) and that \( \sigma_t \) converges to a metric with constant Gauss curvature. Moreover, \( \sigma_t \) has \( K > -\kappa \) for \( t > 0 \).

Let \( I \subset [0, \infty) \) be the set of parameters that \( \sigma_t \) can be isometrically embedded into \( \mathbb{H}^3_{-\kappa} \) as a closed convex \( C^{1,1} \) surface.

2. **Openness:** show that \( I \cap (0, \infty) \) is open.

Since any metric with constant Gauss curvature is the same as the standard metric up to a diffeomorphism, \( \sigma_\infty \) can be isometrically embedded into \( \mathbb{H}^3_{-\kappa} \). As a result of the openness, there exists a sufficiently large \( T_0 \) such that for \( t > T_0 \), \( \sigma_t \) can be isometrically embedded into \( \mathbb{H}^3_{-\kappa} \). In particular, \( I \) is nonempty.

3. **Closedness:** We prove an a priori estimate to obtain closedness.
4.1 Connectedness

We prove the connectedness by using solutions to the normalized Ricci flow.

**Lemma 4.2.** There exists an one-parameter family of smooth metrics $\sigma_t, t \in [0, \infty)$ on $S^2$ such that $\sigma_0 = \bar{\sigma}$ and $\sigma_t$ converges to a metric $\sigma_\infty$ with constant Gauss curvature. Moreover, $K(\sigma_t) > -\kappa$ for $t > 0$. □

**Proof.** For any given smooth metric $\bar{\sigma}$ on $S^2$ with $K(\bar{\sigma}) \geq -\kappa$, consider the normalized Ricci flow with the initial metric $\bar{\sigma}$:

$$\begin{aligned}
\frac{\partial \sigma(t)}{\partial t} &= (r - R(\sigma(t)))\sigma(t) \\
\sigma(0) &= \bar{\sigma},
\end{aligned}$$

where $R(\sigma(t))$ is the scalar curvature of $\sigma(t)$ with average $r = \int_{S^2} R(\sigma(t)) \, d\mu(t) = 8\pi$. Hamilton [9] and Chow [5] established long-time existence and convergence to a metric with constant Gauss curvature. Moreover, the scalar curvature satisfies the evolution equation

$$\frac{\partial R(\sigma(t))}{\partial t} = \Delta_{\sigma(t)} R(\sigma(t)) + R(\sigma(t))(R(\sigma(t)) - r).$$

Applying the strong maximum principle to the evolution equation, we know that $\frac{d}{dt} \min_{S^2} R(\sigma(t)) > 0$ when $\min_{S^2} R(\sigma(t)) < 0$. Since $R(\sigma(t)) = 2K(\sigma(t))$ and $K(\sigma(0)) = K(\bar{\sigma}) \geq -\kappa$, $K(\sigma(t)) + \kappa$ is positive for $t > 0$. $\sigma_t = \sigma(t)$ is the desired family of metrics. ■

4.2 Openness

We show the openness in the continuity method by proving the following result.

**Theorem 4.3.** Let $\sigma$ be a smooth metric on $S^2$ with Gauss curvature $K > -\kappa$. Suppose $\sigma$ can be isometrically embedded into $\mathbb{H}^3_{-\kappa}$ as a closed convex surface $r$. Then for any $\alpha \in (0, 1)$, there exists a positive $\epsilon$, depending only on $\sigma$ and $\alpha$, such that any smooth metric $\sigma'$ on $S^2$ satisfying

$$|\sigma - \sigma'|_{C^{2,\alpha}} < \epsilon$$

can be isometrically embedded in $\mathbb{H}^3_{-\kappa}$ as a closed convex surface $r'$. □

Recall that we write $r$ for $\varphi \circ i$ and view $r$ as the position vector, where $\varphi$ is a trivialization. A deformation $\tau \in \Gamma(S^2, E)$ is viewed as a vector-valued function $y : S^2 \to \mathbb{R}^3$.
and a translation sending \( r \) to \( r + y \) at the same time. To prove Theorem 4.3, it suffices to find a vector \( y \) satisfying
\[
 g_{ij}(r + y) \frac{\partial (r^i + y^i)}{\partial t^a} \frac{\partial (r^j + y^j)}{\partial t^b} = \sigma'_{ab}.
\] (15)

Below, we first find an equivalent infinitesimal-deformation equation (16) and solve the linearized equation (17) of (16).

Subtracting \( g_{ij}(r) \frac{\partial r^i}{\partial t^a} \frac{\partial r^j}{\partial t^b} = \sigma_{ab} \) from (15), we get
\[
 g_{ij}(r)(r^i_a D_b y^j + D_a y^i r^j_b) - g_{ij}(r)(r^i_a \Gamma^j_{bk} y^k + \Gamma^i_{ak} y^j r^j_b) + g_{ij}(r)y^i_a y^j_b
 + (g_{ij}(r + y) - g_{ij}(r))(r^i_a r^j_b + r^j_a r^i_b + y^i_a r^j_b + y^j_a r^i_b) = \sigma'_{ab} - \sigma_{ab},
\]
where \( D_b = D_a \frac{\partial}{\partial t^a} \), \( r^i_a = \frac{\partial r^i}{\partial t^a} \), and \( \Gamma^i_{ak} = \Gamma^i_{mk} \). By Taylor theorem,
\[
 g_{ij}(r + y) - g_{ij}(r) = \partial_k g_{ij}(r) y^k + y^k y^j \int_0^1 (1 - t) \partial^2_k g_{ij}(r + ty) \, dt.
\]

By the definition of Christoffel symbol, \( \partial_k g_{ij}(r) r^i_a r^j_b = g_{ij}(r)(r^i_a r^j_b + \Gamma^i_{ak} r^j_b) \). Set
\[
 F_{ijkl}(r, y) := \int_0^1 (1 - t) \partial^2_k g_{ij}(r + ty) \, dt \quad \text{and} \quad G_{ijkl}(r, y) := \int_0^1 \partial_k g_{ij}(r + ty) \, dt.
\]

We conclude that (15) is equivalent to the following inhomogeneous infinitesimal-deformation-type equation
\[
 g_{ij}(r)(r^i_a D_b y^j + D_a y^i r^j_b)
 = \sigma'_{ab} - \sigma_{ab} - g_{ij}(r) y^i_a y^j_b - F_{ijkl}(r, y) r^i_a r^j_b y^k y^j - G_{ijkl}(r, y) y^k(r^i_a y^j_b + y^i_a r^j_b + y^j_a r^i_b)
 =: q_{ab}(y)
\] (16)

Note that \( |F_{ijkl}(r, y)|_{m, \alpha}, |G_{ijkl}(r, y)|_{m, \alpha} \leq C_{m, \alpha} |y|_{m, \alpha} \).

To solve (16), we study the corresponding linearized equation
\[
 g_{ij}(r)(r^i_a D_b y^j + D_a y^i r^j_b) = \bar{q}_{ab}
\] (17)
where \( \bar{q}_{ab} \) is an arbitrary smooth symmetric bilinear form on \( S^2 \).

Before solving the linearized equation (17), we show that \( \bar{D} \) is a flat connection on \( E \).

**Lemma 4.4.** For any \( Y \in \Gamma(\Sigma, E) \),
\[
 \bar{D}_{e_a} \bar{D}_{e_a} Y = \bar{D}_{e_a} \bar{D}_{e_a} Y - \bar{D}_{[e_a, e_b]} Y = 0. \]

Proof.

\[
\tilde{D}_{ea} \tilde{D}_{eb} Y = \tilde{D}_{eb} \left( D_{ea} Y + \frac{\kappa}{f} (e_a, Y) X \right) \\
= D_{eb} D_{ea} Y + D_{eb} \left( \frac{\kappa}{f} (e_a, Y) X \right) + \frac{\kappa}{f} (e_b, D_{ea} Y) X + \frac{\kappa^2}{f^2} (e_a, Y) (e_b, X) X \\
= D_{eb} D_{ea} Y - \frac{\kappa^2}{f^2} (e_b, X) (e_a, Y) X + \frac{\kappa}{f} (D_{eb} e_a, Y) X + \frac{\kappa}{f} (e_a, D_{eb} Y) X + \kappa (e_a, Y) e_b \\
\quad + \frac{\kappa}{f} (e_b, D_{ea} Y) X + \frac{\kappa^2}{f^2} (e_a, Y) (e_b, X) X.
\]

Therefore,

\[
\tilde{D}_{ea} \tilde{D}_{ea} Y - \tilde{D}_{ea} \tilde{D}_{eb} Y - \tilde{D}_{eb} \tilde{D}_{ea} Y = R(e_b, e_a) Y + \kappa (e_a, Y) e_b - \kappa (e_b, Y) e_a = 0. \quad \square
\]

Next, we solve the linearized equation \((17)\).

**Proposition 4.5.** For any smooth symmetric bilinear form \(\tilde{q}\) on a convex surface \(\Sigma \subset \mathbb{H}^3_{-\kappa}\), there exists a smooth solution to

\[
D \tau \odot DX = f^2 \tilde{q}.
\] (18)

**Proof.** For a fixed point \(p\), we choose an orthonormal frame \(\{e_1, e_2\}\) with \(D^T_{ea} e_b(p) = 0\) and write \(D_a\) for \(D_{ea}\) and \(\tilde{q}_{ab}\) for \(\tilde{q}(e_a, e_b)\). Equation \((18)\) implies the symmetric part of tangential components of \(D \tau\) is \(f \tilde{q}_{ab}\), equivalently,

\[
\frac{1}{2} \left( \tilde{D}_a \left( \frac{\tau}{f} \right) \cdot e_b + \tilde{D}_b \left( \frac{\tau}{f} \right) \cdot e_a \right) = \tilde{q}_{ab}. \quad (19)
\]

To solve \((18)\), we introduce new dependent variables. We define \(v_1, v_2,\) and \(w\) by

\[
v_a = \tilde{D}_a \left( \frac{\tau}{f} \right) \cdot \nu, \quad a = 1, 2. \quad (20)
\]

and

\[
\frac{w}{f^2} = \frac{1}{2} \left( \tilde{D}_a \left( \frac{\tau}{f} \right) \cdot e_b - \tilde{D}_b \left( \frac{\tau}{f} \right) \cdot e_a \right) \quad (21)
\]

The triplet \(\{v_1, v_2, w\}\) completely determines \(\tilde{D}(\frac{\tau}{f})\):

\[
\tilde{D}_a \left( \frac{\tau}{f} \right) = \sum_{b=1}^2 \left( \tilde{q}_{ab} + \frac{w}{f^2} \epsilon_{ab} \right) e_b + v_a \nu \quad (22)
\]
where $\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let $\nabla$ denote the Levi–Civita connection with respect to $\sigma$. Since $\tilde{D}$ is a flat connection, we have

\[
0 = \tilde{D}_a \tilde{D}_b \left( \frac{\tau}{f} \right) - \tilde{D}_b \tilde{D}_a \left( \frac{\tau}{f} \right) = \sum_{c=1}^{2} \left[ \nabla_a \left( \tilde{q}_{bc} + \frac{w}{f^2} \epsilon_{bc} \right) e_c - \left( \tilde{q}_{bc} + \frac{w}{f^2} \epsilon_{bc} \right) h_{ac} v + v_b h_{ac} e_c \right] + \frac{\kappa}{f} \left( \tilde{q}_{ba} + \frac{w}{f^2} \epsilon_{ba} \right) X - (a, b \text{ antisymmetric}).
\]

Comparing the tangential and normal components, we have

\[
\begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix} \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix} = \begin{pmatrix} \frac{\nabla_1 w}{f^2} - c_1 \\ \frac{\nabla_2 w}{f^2} - c_2 \end{pmatrix}, \tag{23}
\]

and

\[
\nabla_1 v_2 - \nabla_2 v_1 = T - \frac{H w}{f^2} + \frac{2\kappa}{f^3} w \langle X, v \rangle, \tag{24}
\]

where $T = -\sum_{a=1}^{2} (\tilde{q}_1 a h_{a2} + \tilde{q}_2 a h_{a1})$ and $c_a = \nabla_1 \tilde{q}_{a2} - \nabla_2 \tilde{q}_{a1}$. We substitute (23) into (24) to obtain an elliptic equation

\[
\nabla_a \left( (h^{-1})_{ab} \frac{\nabla_b w}{f^2} \right) + \frac{H w}{f^2} - \frac{2\kappa}{f^3} \langle X, v \rangle w = T + \nabla_a ((h^{-1})_{ab} c_b). \tag{25}
\]

as $\det(h_{ab}) > 0$.

To solve the self-adjoint elliptic equation (25), we need to show the quantity of the right-hand side is perpendicular to the kernel of the operator. The homogeneous equation associated with (25) corresponds to an infinitesimal isometric deformation. From the infinitesimal rigidity result in Section 3, the general solution comes from Killing vector fields $Y_0 \times X + fZ_0$.

For $\tau = Y_0 \times X + fZ_0$, by (3.7),

\[
w = \frac{1}{2} f^2 \left( \tilde{D}_a \left( \frac{\tau}{f} \right) \cdot e_2 - \tilde{D}_a \left( \frac{\tau}{f} \right) \cdot e_1 \right) = \langle Y_0, v \rangle + \kappa \langle Y_0, X \rangle \langle X, v \rangle.
\]

Hence the kernel is of the form

\[
\bar{w} = \langle Y_0, v \rangle + \kappa \langle Y_0, X \rangle \langle X, v \rangle
\]
for any constant vector \( Y_0 \). To show
\[
\int T \bar{w} - (h^{-1})^{ab} c_b \nabla_a \bar{w} = 0,
\]
we introduce new quantities
\[
\bar{w}_a = (Y_0, e_a) + \kappa (Y_0, X) (X, e_a), \quad a = 1, 2.
\]
Since \( \bar{D} Y_0 = 0 \), we have the following
\[
\nabla_a \bar{w} = \sum_{b=1}^{2} h_{ab} \bar{w}_b,
\]
and
\[
\nabla_a \bar{w}_b = -h_{ab} \bar{w} + \kappa f(Y_0, X) \delta_{ab}.
\]
We are now ready to verify that
\[
\int T \bar{w} - (h^{-1})^{ab} c_b \nabla_a \bar{w} = \int T \bar{w} - \sum_{a=1}^{2} c_a \bar{w}_a
\]
\[
= \int T \bar{w} + \sum_{a=1}^{2} (\bar{q}_{a2} \nabla_1 \bar{w}_a - \bar{q}_{a1} \nabla_2 \bar{w}_a)
\]
\[
= \int T \bar{w} - \sum_{a=1}^{2} \bar{q}_{a2} h_{1a} \bar{w} + \bar{q}_{12} \kappa f(Y_0, X) + \sum_{a=1}^{2} \bar{q}_{a1} h_{2a} \bar{w} - \bar{q}_{21} \kappa f(Y_0, X)
\]
\[
= 0.
\]
From Hilbert’s theory, \( w \) can be solved for (25). By the regularity theory for elliptic equations, \( w \) is smooth. We then solve \( u_1 \) and \( u_2 \) from \( w \) in (23). At last, choose a point \( p \) and initial value \( \tau(p) \) and integrate (22) along paths to get \( \tau \).

We are in the position to prove Theorem 4.3.

**Proof of Theorem 4.3.** Given a vector-valued function \( z \), let \( y = \phi(z) \) be the solution of (18) with the right-hand side \( \bar{q} = q_{ab}(z) \). Note that \( y \) solves (15) if \( y \) is a fixed point of \( \phi \). We intend to apply the contraction mapping principle to find a fixed point. First of all, we need an a priori estimate of the solution of (18).
Lemma 4.6 ([10, Lemma 9.2.4]). Given $0 < \alpha < 1$, and $z \in T_{r} \mathbb{R}^3$, there exist a smooth solution $y$ of (18) and a constant $C$ depending on $\alpha$ and $\Sigma$ such that

$$|y|_{2,\alpha} \leq C \left( \left| \frac{q(z)}{f} \right|_{1,\alpha} + |\nabla_1 (f^2 c_2) - \nabla_2 (f^2 c_1)|_{\alpha} \right)$$

$$\leq C (|q(z)|_{1,\alpha} + |\nabla_1 c_2 - \nabla_2 c_1|_{\alpha}).$$

Here $c_a = \nabla_1 (q(z)_{a2}) - \nabla_2 (q(z)_{a1})$ is defined as in the proof of the previous lemma. \hfill \Box

Observe that $\nabla_1 (f^2 c_2) - \nabla_2 (f^2 c_1)$ does not involve the third derivatives of $z$ and every term contains at least two $z$'s. Lemma 4.6 implies

$$|\phi(z)|_{2,\alpha} \leq C_1 (|\sigma_{ab} - \sigma_{ab}|_{2,\alpha} + |z|_{2,\alpha}^2).$$

Note that the solution is linear in $q_{ab}$. Thus, if $y$ is the solution of $g_{ij}(r)(r_i^a D_b y^j + D_a y^i r_b^j) = q_{ab}(z)$, and $y'$ is the solution of $g_{ij}(r)(r_i^a D_b y'^j + D_a y'^i r_b^j) = q_{ab}(w)$, then the difference $y - y'$ satisfies the equation with right-hand side $\tilde{q} = q(z) - q(w)$. Hence, we have

$$|y - y'|_{2,\alpha} \leq C (|\tilde{q}|_{1,\alpha} + |\nabla_1 c_2 - \nabla_2 c_1|_{\alpha}).$$

where $c_1, c_2$ are expressed in terms of the coefficients of $\tilde{q}$. The term $|\nabla_1 c_2 - \nabla_2 c_1|_{\alpha}$ does not involve the derivatives of $y$ and $y'$ of order higher than two. Given a symmetric bilinear form $A_{ij}$ and two sections $\alpha, \beta \in \bigcap (S^2, E \otimes T^* S^2)$, define $A(\alpha \cdot \beta) = \frac{1}{2} A_{ij}(\alpha^i \beta^j + \alpha^j \beta^i)$. Note that $A(dz \cdot dz - dw \cdot dw) = A(d(z + w) \cdot d(z - w))$ for any $A$. Hence we have

$$q(z) - q(w) = -g(d(z + w) \cdot dz - dw \cdot dw)$$

$$- \left[ (z - w)^m \left( \int_0^1 \frac{\partial F_{ijkl}}{\partial y^m} (tz + (1 - t)w) \ dt \right) z^k z^l \right]$$

$$+ F_{ijkl}(w)(z + w)^k(z - w)^l r_i^r r_j^s$$

$$- \left[ (z - w)^m \left( \int_0^1 \frac{\partial G_{ijkl}}{\partial y^m} (tz + (1 - t)w) \ dt \right) z^k (2 \ dr \cdot dz + dz \cdot dz) \right]$$

$$+ G_{ijk}(w)(z - w)^k(2 \ dr \cdot dz + dz \cdot dz)$$

$$+ G_{ijk}(w)w^k(2 \ dr \cdot d(z - w) + d(z + w) \cdot d(z - w))$$

If $|z|_{2,\alpha}, |w|_{2,\alpha} < 1$, then

$$|\phi(z) - \phi(w)|_{2,\alpha} \leq C_2 (|z|_{2,\alpha} + |w|_{2,\alpha}) z - w|_{2,\alpha}.$$
If we choose \( \mu < 1 \) such that \( C_1 \mu < \frac{1}{2} \) and \( 2C_2 \mu < 1 \), then for any metric \( \sigma' \) with \( C_1 |\sigma' - \sigma|_{2,\alpha} < \mu \), \( \phi : B_{\mu} \to B_{\mu} \) is a contraction mapping in \( C^{2,\alpha} \). The existence of solution to (16) follows from contraction mapping principle. This completes the proof of Theorem 4.3.

From Theorem 4.3 and Lemma 4.2, \( I \) is open and nonempty as \( (T_0, \infty) \subset I \) for some large \( T_0 \).

### 4.3 Closedness

To prove closedness, we have to establish the a priori estimate for the isometric embedding. Suppose we have a sequence of isometric embeddings \( r_t \) with \( t_i \to T \). Recall that we fix a diffeomorphism \( \varphi : \mathbb{H}^3_{-\kappa} \to \mathbb{R}^3 \).

**Definition 4.7.** We say that a surface \( \Sigma \subset \mathbb{H}^3_{-\kappa} \) is centered at the origin if \( \varphi(\Sigma) \) has center of mass at \((0, 0, 0)\).

By an isometry in \( \mathbb{H}^3_{-\kappa} \), we may assume that the embeddings \( r_t \) are centered at the origin.

In 2D, the Ricci flow equation (14) can be rewritten as a parabolic equation of a scalar function. By the uniformization theorem, \( \bar{\sigma} = e^{2\bar{u}} \hat{\sigma} \) for a metric \( \hat{\sigma} \) with constant Gauss curvature \( \hat{K} \) and the same area as \( \bar{\sigma} \). Let \( \sigma_t = e^{2u} \hat{\sigma} \), then (14) becomes an equation of \( u \)

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \hat{K} - K \\
u_0 &= \bar{u}
\end{aligned}
\]  

(26)

Moreover, by the work of M. Struwe, we have the following theorem.

**Theorem 4.8** ([19, Theorem 6.1]). For any \( u_0 \in H^2(S^2, \hat{\sigma}) \), there exist a unique global solution \( u \) of (26) and a smooth limit \( u_\infty \) corresponding to a smooth metric \( \sigma_\infty = e^{2u_\infty} \hat{\sigma} \) of constant curvature such that

\[
\|u(t) - u_\infty\|_{H^2} \leq Ce^{-\alpha t}
\]  

(27)

for some constants \( C \) and \( \alpha \) depending only on \( \hat{\sigma} \) and \( u_0 \).

From (27), the diameters of \( \sigma_t \) are uniformly bounded. Thus, \( \max_{\Sigma_t} r^i \) and \( \max_{\Sigma_t} f \) are bounded by a constant depending only on \( \bar{\sigma} \). This proves the uniform \( C^0 \)-estimate for \( r_t \).
The $C^1$-estimate follows from the isometric embedding equation

$$g_{ij} \frac{\partial r^i}{\partial u^i} \frac{\partial r^j}{\partial u^j} = \sigma_{ab}.$$

Indeed, we may assume $g_{ij}$ is diagonal everywhere by a change of variable. Then for $i = 1, 2, 3$, we can directly check

$$|\nabla r^i|^2 = \sigma^{ab} r_a^i r_b^i \leq C g_{jk} \sigma^{ab} r_a^j r_b^k = 2C,$$

where $C$ only depends on $\max_{\Sigma} f$.

In the following, we write $\sigma$ for $\sigma_t$ and $r$ for $r_t$ if there is no risk of confusion. The key to prove $C^2$-estimate is a uniform bound of the principal curvatures. Since $\Sigma$ is convex, it suffices to bound the mean curvature. The main difficulty lies in that the Gauss curvature of a convex surface in hyperbolic space may be negative somewhere. We remark that for the strictly convex case, the uniform bound of the mean curvature is proved by Pogorelov [18, pp. 337–342].

**Theorem 4.9 (Theorem B).** Let $\Sigma$ be a closed convex surface in $\mathbb{H}^3_\kappa$, normalized so that $\Sigma$ is centered at the origin. Then

$$\max_{\Sigma} H \leq C,$$

for some constant $C$ depending only on $\| f \|_{C^0(\Sigma)}$ and $\| K \|_{C^2(\Sigma)}$. \hfill $\square$

**Proof.** Let $\lambda \geq \mu$ be the two principal curvatures. We intend to apply the maximum principle to the test function $F = \log \lambda + a \frac{|X|^2}{2}$, where $\alpha$ satisfies

$$\alpha \min_{\Sigma} f^2 > \kappa.$$

Suppose $F$ achieves its maximum at $p$. We may assume that $\lambda > 2\mu$ at $p$; otherwise $\lambda^2 \leq 2\lambda \mu = 2(K + \kappa)$ and the estimate clearly holds. In the following computation, we denote the covariant derivative with respect to $\sigma$ by ; or $\nabla$. Moreover, we write $\lambda_a$ for $\lambda \cdot a$ for the gradient of principal curvatures. The first and second derivatives of $F$ are given by

$$F_a = \frac{\lambda_a}{\lambda} + \alpha f(X, e_a)$$

(28)

and

$$F_{ab} = \frac{\lambda_{ab}}{\lambda} - \frac{\lambda_a \lambda_b}{\lambda^2} + \kappa \alpha (X, e_a)(X, e_b) + \alpha \frac{f^2}{2} \sigma_{ab} - \alpha f h_{ab}(X, v).$$

(29)
We compute each term in \((H\sigma^{ab} - h^{ab})F_{;ab}\). Starting with \((H\sigma^{ab} - h^{ab})\lambda_{;ab}\). We have

\[(H\sigma^{ab} - h^{ab})\lambda_{;ab} = \mu\lambda_{;11} + \lambda\lambda_{;22}.\]

By (A.2) and the Codazzi equation,

\[(H\sigma^{ab} - h^{ab})\lambda_{;ab} = \mu\left(h_{11;11} + \frac{2}{\lambda - \mu}(h_{11;2})^2\right) + \lambda\left(h_{11;22} + \frac{2}{\lambda - \mu}(h_{22;1})^2\right).\]

By the Codazzi equation and commutation formula, \(h_{11;22} = h_{22;11} + K(\lambda - \mu)\). Thus,

\[(H\sigma^{ab} - h^{ab})\lambda_{;ab} = \mu\left(h_{11;11} + \frac{2}{\lambda - \mu}(\lambda_2)^2\right) + \lambda\left(h_{22;11} + \frac{2}{\lambda - \mu}(\mu_1)^2\right) + K\lambda(\lambda - \mu).\]

On the other hand, differentiating the Gauss equation \(\det(h) = K + \kappa\), we get

\[(K + \kappa)_{;ab} = ((H\sigma^{cd} - h^{cd})h_{cd;a})_{;b}\]

\[= (H\sigma^{cd} - h^{cd})h_{cd;ab} + H_aH_b - h^{cd}_{;b}h_{cd;a}.\]

In particular,

\[K_{;11} = \mu h_{11;11} + \lambda h_{22;11} + (\lambda_1 + \mu_1)^2 - (\lambda_1)^2 - 2(\lambda_2)^2 - (\mu_1)^2\]

\[= \mu h_{11;11} + \lambda h_{22;11} + 2\lambda_1\mu_1 - 2(\lambda_2)^2.\]  

Therefore,

\[(H\sigma^{ab} - h^{ab})\lambda_{;ab} = K_{;11} - 2\lambda_1\mu_1 + \frac{2\lambda}{\lambda - \mu}((\lambda_2)^2 + (\mu_1)^2) + K\lambda(\lambda - \mu).\]  

At \(p\), the derivatives \(F_{;a} = 0\) and \(F_{;ab} \leq 0\). We thus have

\[\lambda_1 = -\alpha f(X, e_1)\lambda,\]

\[\lambda_2 = -\alpha f(X, e_2)\lambda,\]

\[\mu_1 = \frac{K_1}{\lambda} + \frac{\alpha f(K + \kappa)(X, e_1)}{\lambda} = O(1/\lambda),\]

\[\mu_2 = \frac{K_2}{\lambda} - \frac{\alpha f(K + \kappa)(X, e_2)}{\lambda} = O(1/\lambda);\]
and

\[
0 \geq (H\sigma_{ab} - h_{ab})F_{ab} = \frac{1}{\lambda} \left( K_{11} - 2\lambda\mu_1 + \frac{2\lambda}{\lambda - \mu}((\lambda_2)^2 + (\mu_1)^2) + K\lambda(\lambda - \mu) \right) - \alpha^2 f^2 \langle X, e_1 \rangle^2 \mu - \alpha^2 f^2 \langle X, e_2 \rangle^2 \lambda \\
+ \kappa \alpha (\mu \langle X, e_1 \rangle^2 + \lambda \langle X, e_2 \rangle^2) + \alpha f^2 (\lambda + \mu) - 2\alpha f(K + \kappa)\langle X, \nu \rangle \\
= (2\alpha^2 f^2 \langle X, e_2 \rangle^2 + K - \alpha^2 f^2 \langle X, e_2 \rangle^2 + \kappa \alpha \langle X, e_2 \rangle^2 + \alpha f^2)\lambda + O(1) \\
\geq (K + \alpha f^2)\lambda + O(1).
\]

Note that the first term in the last equality comes from \((\lambda_2)^2\). Here, we say a function \(G = O(\lambda^p)\) if there exist some constants \(c\) and \(C\) depending only on \(\|K\|_{C^2(\Sigma)}\) and \(\|f\|_{C^0(\Sigma)}\) such that \(c\lambda^p \leq G \leq C\lambda^p\) when \(\lambda \geq 1\).

From our assumption on \(\alpha\), \(\lambda(p) \leq C\). For other points \(q \in \Sigma\),

\[
\lambda(q) \leq \lambda(p) \frac{e^{\|X\|^2(p)}}{e^{\|X\|^2(q)}} \leq C.
\]

We are in the position to prove the \(C^2\)-estimate. Writing \(D_{r^i}r_a\) in two ways

\[
(D_{r^i}r_a)^i = \frac{\partial^2 r^i}{\partial u^a \partial u^b} + \Gamma^i_{jk} \frac{\partial r^j}{\partial u^a} \frac{\partial r^k}{\partial u^b} \\
= \Gamma^i_{ab} \frac{\partial r^a}{\partial u^c} - h_{ab} v^i,
\]

we obtain

\[
\nabla_b \nabla_a r^i = -h_{ab} v^i - \Gamma^i_{jk} \frac{\partial r^j}{\partial u^b} \frac{\partial r^k}{\partial u^a}. \tag{33}
\]

Hence \(\|r^i\|_{C^2} \leq C\) where \(C\) depends on the upper bound of principal curvatures and \(\|r^i\|_{C^1}\). By Arzela–Ascoli theorem, a subsequence of \(r_k\) converges to some \(r_T \in C^{1,1}\). This completes the proof of Theorem 4.1.

When \(K > -\kappa\), the continuity method actually produces a smooth isometric embedding.

**Theorem 4.10.** Let \(\tilde{\sigma}\) be a smooth metric on \(S^2\) with Gauss curvature \(K > -\kappa\). Then there exists a smooth isometric embedding \(i : (S^2, \tilde{\sigma}) \rightarrow \mathbb{H}^{3}_{-\kappa}\), which is unique up to congruence.
Proof. The proof of uniqueness (independent of Pororelov’s) could be found in [8]. To prove the theorem, we have to establish a priori estimates for the higher derivatives of \( r_t \). Let \( \Sigma_t \) denote \( r_t(\Sigma) \). Define

\[
\rho(t) = \frac{1}{2} (X|_{\Sigma_t}, X|_{\Sigma_t}).
\]

We compute

\[
\rho_{,a} = f \left< X, \frac{\partial r}{\partial u^a} \right>,
\]

\[
\rho_{,ab} = \kappa \left< X, \frac{\partial r}{\partial u^a} \right> \left< X, \frac{\partial r}{\partial u^b} \right> + f \sigma_{ab} - h_{ab}\langle X, v \rangle.
\]

(34)

Taking the determinant of (34), we get

\[
F \equiv \det(\rho_{,ab} - f \sigma_{ab} - (K + \kappa)(2\rho - f^2|\nabla \rho|^2)) = 0
\]

(35)

The assumption \( K > -\kappa \), together with Lemmas 4.11 and 4.12, imply that (35) is uniformly elliptic

\[
F_{\rho_{,11}}F_{\rho_{,22}} - F_{\rho_{,12}}^2 = \frac{K + \kappa}{\det \sigma} (X, v)^2 > 0.
\]

By [14, Theorem I] (see also [10, Lemma 9.3.4]) and the Schauder estimates, \( \|\rho\|_{C^{m,\alpha}} \) is uniformly bounded for any \( m \) and \( 0 < \alpha < 1 \). The higher regularity of \( r \) follows from (33) and (34).

By (34), since the support function \( \langle X, v \rangle \) is bounded from above and below, the \( m \)th derivatives of second fundamental form \( \nabla^m h \) are bounded by the \( (m + 2) \)th derivatives of \( \rho \) and \( (m + 1) \)th derivatives of \( r \). Furthermore, by (33), the \( (m + 2) \)th derivatives of \( r \) is bounded by the \( m \)th derivatives of \( h \) and the \( (m + 1) \)th derivatives of \( r \). The higher regularity of \( r \) follows. This completes the proof.

\[\square\]

Lemma 4.11. Suppose that \( \Sigma \) is a smooth closed convex surface centered at the origin in \( H^3_{-\kappa} \). Then there exists a positive constant \( R \) depending on \( \max K + \kappa \) and the diameter of \( \Sigma \) such that the geodesic ball of radius \( R \) at the origin lies inside \( \Sigma \).

Proof. We identify the hyperbolic space \( H^3_{-\kappa} \) with the hyperboloid \( (x^0, x^1, x^2, x^3) : -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -\frac{1}{\kappa} \). Consider the Beltrami map

\[
\beta : H^3_{-\kappa} \to \left\{ x^0 = \frac{1}{\sqrt{\kappa}} \right\}
\]
We identify \( \{x^0 = \frac{1}{\sqrt{\kappa}} \} \) with the Euclidean space \( \mathbb{R}^3 \). Recall the static potential \( f \) is equal to \( \sqrt{\kappa} x^0 \). Suppose \( \Sigma \) is given by the embedding
\[
\mathbf{r}(u^1, u^2) = (x^0(u^1, u^2), x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2))
\]
with metric
\[
\sigma_{ab} = -\frac{1}{\kappa} \frac{\partial f}{\partial u^a} \frac{\partial f}{\partial u^b} + \sum_{i=1}^{3} \frac{\partial x^i}{\partial u^a} \frac{\partial x^i}{\partial u^b},
\]
then the embedding of \( \tilde{\Sigma} = \beta(\Sigma) \) is given by \( \frac{\mathbf{r}}{f} - \frac{\partial}{\partial x^0} \) where \( \frac{\partial}{\partial x^0} = (1, 0, 0, 0) \). We compute the induced metric of \( \tilde{\Sigma} \)
\[
\tilde{\sigma}_{ab} = \left( \frac{1}{f^2} \frac{\partial \mathbf{r}}{\partial u^b} - \frac{1}{f^2} \frac{\partial f}{\partial u^b} \mathbf{r} - \frac{1}{f^2} \frac{\partial f}{\partial u^b} \right)
\]
\[
= \frac{1}{f^2} \left( \sigma_{ab} - \frac{\frac{\partial f}{\partial u^a} \frac{\partial f}{\partial u^b}}{\kappa f^2} \right).
\]
It is not hard to check that the unit normals of \( \Sigma \) and \( \tilde{\Sigma} \) are related by
\[
\tilde{\nu} = \nu + \left( \nu, \frac{\partial}{\partial x^0} \right) \frac{\partial}{\partial x^0} \sqrt{1 + \nu, \frac{\partial}{\partial x^0})^2}
\]
Next we compute the second fundamental forms of \( \Sigma \) and \( \tilde{\Sigma} \)
\[
h_{ab} = -\left( \frac{\partial^2 \mathbf{r}}{\partial u^a \partial u^b}, \nu \right)
\]
\[
\tilde{h}_{ab} = -\left( \frac{\partial}{\partial u^b} \left( \frac{1}{f} \frac{\partial \mathbf{r}}{\partial u^a} - \frac{1}{f^2} \frac{\partial f}{\partial u^a} \mathbf{r} \right), \nu \right) \left( 1 + \nu, \frac{\partial}{\partial x^0})^2\right)^{-1}
\]
\[
= \frac{1}{\sqrt{1 + \nu, \frac{\partial}{\partial x^0})^2 f}} h_{ab}.
\]
Hence \( \tilde{\Sigma} \) is convex. The Gauss curvatures of \( \Sigma \) and \( \tilde{\Sigma} \) are related by
\[
\tilde{K} = \frac{\det(h_{ab})}{\det(\tilde{\sigma}_{ab})}
\]
\[
= \frac{f^2}{\left( 1 + \nu, \frac{\partial}{\partial x^0} \right)^2 (1 - \frac{\left| \nu \mathbf{f} \right|^2}{f^2}) \left( K + \kappa \right)}
\]
From (4), we have
\[
1 - \frac{\|\nabla f\|^2}{\kappa f^2} = \frac{1 + \kappa \langle X, \nu \rangle}{f^2}
\]
Hence \( K \leq (\max K + \kappa)(\max f)^4 \). We apply [10, Lemma 9.1.1] to conclude that there exists a positive constant \( R \) depending only on \( \frac{1}{\max K} \) and the diameter of \( \tilde{\Sigma} \) such that there exists a ball of radius \( R \) inside \( \tilde{\Sigma} \). Therefore, there exists a positive constant \( R' \) depending only on \( \frac{1}{\max K + \kappa} \) and the diameter of \( \Sigma \) such that there exists a ball of radius \( R' \) inside \( \Sigma \).

**Lemma 4.12.** For a convex surface \( \Sigma \) in \( \mathbb{H}^3_{-\kappa} \), \( \min_{\Sigma} \langle X, \nu \rangle = \min_{\Sigma} r \). □

**Proof.** At the critical points of \( \langle X, \nu \rangle \),
\[
0 = \nabla_a \langle X, \nu \rangle = h(X^T, \cdot).
\]
Since \( \Sigma \) is convex, we have \( X^T = 0 \). □

5 Isometric Embeddings into the Anti-de Sitter Spacetime

We consider the isometric embedding problem of \((S^2, \sigma)\) into the anti-de Sitter spacetime \((AdS, g_{AdS})\). We work on the cosmological chart of \( AdS \) on which the metric \( g_{AdS} \) can be expressed as
\[
g_{AdS} = -dt^2 + S^2(t)g.
\]
where \( S(t) = \cos(\sqrt{\kappa} t) \) and \( g \) is the hyperbolic metric with sectional curvature \( -\kappa \) [11, (5.9)].

We have the following theorem.

**Theorem 5.1** (Theorem C). Given a smooth metric \( \sigma \) and a smooth function \( s \) on \( S^2 \). Suppose
\[
K + \frac{S'}{S} \Delta s - \left( \frac{S'}{S} - \frac{S^2}{S^2} \right) |\nabla s|^2 + (1 + |\nabla s|^2)^{-1} \left( \frac{\det(\nabla^2 s)}{\det \sigma} - \frac{S'}{S} \nabla^a s \nabla^b \nabla_a \nabla_b s \right) > -\kappa. \tag{36}
\]
where \( \nabla \) and \( \Delta \) denote the gradient and Laplace operator with respect to \( \sigma \). Then there exists a unique space-like isometric embedding \( r : (S^2, \sigma) \to AdS \) with prescribed cosmological time function \( s \). □
Proof. Suppose we have an isometric embedding into the cosmological chart \( \mathbf{r} = (s, r^1, r^2, r^3) : (S^2, \sigma) \to \text{AdS} \). Composing \( \mathbf{r} \) with the projection on to the \( \{ t = 0 \} \)-slice, we get another embedding \( \hat{\mathbf{r}} : S^2 \to \mathbb{H}^3_{-\kappa} \), where \( \hat{\mathbf{r}} = (r^1, r^2, r^3) \). The induced metric \( \hat{\sigma} \) of \( \hat{\mathbf{r}} \) in \( \mathbb{H}^3_{-\kappa} \) satisfies
\[
\hat{\sigma} = S^{-2}(s)(ds^2 + \sigma) = S^{-2}(s)\bar{\sigma},
\]
where \( \bar{\sigma} = ds^2 + \sigma \).

Denote the Gauss curvatures of \( \sigma, \hat{\sigma}, \) and \( \hat{\sigma} \) by \( K, \tilde{K}, \) and \( \check{K} \), respectively. The Gauss curvatures \( \tilde{K} \) and \( \check{K} \) can be computed as
\[
\tilde{K} = S^2(s)(\tilde{K} + \Delta_{\hat{\sigma}} \ln S(s)),
\]
\[
\check{K} = (1 + |\nabla s|^2)^{-1} \left( K + (1 + |\nabla s|^2)^{-1} \frac{\det(\nabla^2 s)}{\det \sigma} \right),
\]
where \( \Delta_{\hat{\sigma}} \) is the Laplacian with respect to \( \hat{\sigma} \) and \( \nabla^2 \) denotes the Hessian with respect to \( \sigma \). Moreover,
\[
\Delta_{\hat{\sigma}} \ln S(s) = \left( \sigma^{ab} - \frac{\nabla^a s \nabla^b s}{1 + |\nabla s|^2} \right) \left( \frac{S'}{S} \cdot \frac{\nabla_a \nabla_b s}{1 + |\nabla s|^2} - \left( \frac{S''}{S} - \frac{S'}{S^2} \right) \nabla_a s \nabla_b s \right)
\]
\[
= \frac{S'}{S} \cdot \frac{\Delta s}{1 + |\nabla s|^2} - \left( \frac{S'}{S} - \frac{S^2}{S^2} \right) \frac{|\nabla s|^2}{1 + |\nabla s|^2} - \frac{S'}{S} \cdot \frac{\nabla^a s \nabla^b s \nabla_a \nabla_b s}{(1 + |\nabla|^2)^2}.
\]

Thus,
\[
\check{K} = \frac{S^2(s)}{(1 + |\nabla s|^2)^2} \left\{ K + \frac{S'}{S} \Delta s - \left( \frac{S'}{S} - \frac{S^2}{S^2} \right) |\nabla s|^2 \right.
\]
\[
+ (1 + |\nabla s|^2)^{-1} \left( \frac{\det(\nabla^2 s)}{\det \sigma} - \frac{S'}{S} \nabla^a s \nabla^b s \nabla_a \nabla_b s \right) \right\}.
\]

The above computation shows that if (36) holds, then by Theorem 4.10, there exists an isometric embedding \( (r^1, r^2, r^3) : (S^2, \hat{\sigma}) \to \mathbb{H}^3_{-\kappa} \) and \( (s, r^1, r^2, r^3) : (S^2, \sigma) \to \text{AdS} \) is the desired isometric embedding into anti-de Sitter spacetime.

As for the uniqueness, we assume that \( \mathbf{r}_a = (s, r^1_a, r^2_a, r^3_a) : (S^2, \sigma) \to \text{AdS} \), \( a = 1, 2 \) are two isometric embeddings. Since the induced metrics of the projection \( \hat{\mathbf{r}}_1 \) and \( \hat{\mathbf{r}}_2 \) are isometric, they differ by an isometry in \( \mathbb{H}^3_{-\kappa} \). Consequently, \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) are congruent. ■

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Appendix. Derivatives of Eigenvalues

The purpose of this appendix is to prove a special case of the following well-known fact [1, Theorem 5.5].

**Proposition A.1.** Let $M : \mathbb{R}^m \to \{\text{symmetric } n \times n \text{ matrices}\}$ be a smooth matrix-valued function with distinct eigenvalues $\lambda_1(x), \ldots, \lambda_n(x)$. Suppose $M(0)$ is diagonal. Then we have

\[
\frac{\partial \lambda_i}{\partial x^a}(0) = \frac{\partial M_{ii}}{\partial x^a}(0) \quad (A.1)
\]

\[
\frac{\partial^2 \lambda_i}{\partial x^a \partial x^b}(0) = \frac{\partial^2 M_{ii}}{\partial x^a \partial x^b}(0) - 2 \sum_{j \neq i} \frac{\partial M_{ij}}{\partial x^a}(0) \frac{\partial M_{ij}}{\partial x^b}(0) \lambda_j(0) - \lambda_i(0) \quad (A.2)
\]

\[\square\]

**Proof.** Denote the adjoint matrix of $M$ by $M^*$. By definition,

\[
0 = \det(M(x) - \lambda_i(x)I) \quad (A.3)
\]

Differentiating (A.3), we get

\[
0 = \text{Tr} \left( (M - \lambda_i I)^* \left( \frac{\partial M}{\partial x^a} - \frac{\partial \lambda_i}{\partial x^a} I \right) \right).
\]

Since $M(0)$ is diagonal, the only nonzero entry of $(M(0) - \lambda_i(0)I)^*$ is

\[
(M(0) - \lambda_i(0)I)_{ii}^* = \prod_{k \neq i} (\lambda_k - \lambda_i)(0) \quad (A.4)
\]

From (A.4), the first statement follows. To prove the second statement, we differentiate (A.3) twice to get

\[
0 = \text{Tr} \left( \frac{\partial (M - \lambda_i I)^*}{\partial x^b} \left( \frac{\partial M}{\partial x^a} - \frac{\partial \lambda_i}{\partial x^a} I \right) + (M - \lambda_i I)^* \left( \frac{\partial^2 M}{\partial x^a \partial x^b} - \frac{\partial^2 \lambda_i}{\partial x^a \partial x^b} I \right) \right).
\]

By differentiating the equation $(M - \lambda_i I)^*(M - \lambda_i I) = 0$, we observe that the only nonzero entries of $\frac{\partial (M - \lambda_i I)^*}{\partial x^b}(0)$ are

\[
\left[ \frac{\partial (M - \lambda_i I)^*}{\partial x^b}(0) \right]_{ij} = (-1) \left( \prod_{k \neq i} (\lambda_k(0) - \lambda_i(0)) \right) \frac{\partial M_{ij}}{\partial x^b}(0) \frac{\lambda_j(0) - \lambda_i(0)}{\lambda_j(0) - \lambda_i(0)} \quad \text{for } j \neq i. \quad (A.5)
\]
From (A.5), we obtain

\[
0 = \left( \prod_{k \neq i} (\lambda_k(0) - \lambda_i(0)) \right) \left( 2 \sum_{j \neq i} (-1) \frac{\partial^2 M_{ij}(0)}{\partial x^b} \frac{\partial M_{ij}(0)}{\partial x^a} - \frac{\partial^2 \lambda_i(0)}{\partial x^b} \partial x^a \right)
\]

This proves the second statement.

References


