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INVARIANT FOLIATIONS NEAR NORMALLY HYPERBOLIC INVARIANT MANIFOLDS FOR SEMIFLOWS

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ABSTRACT. Let M be a compact C^1 manifold which is invariant and normally hyperbolic with respect to a C^1 semiflow in a Banach space. Then in an ϵ-neighborhood of M there exist local C^1 center-stable and center-unstable manifolds $W^{cs}(\epsilon)$ and $W^{cu}(\epsilon)$, respectively. Here we show that $W^{cs}(\epsilon)$ and $W^{cu}(\epsilon)$ may each be decomposed into the disjoint union of C^1 submanifolds (leaves) in such a way that the semiflow takes leaves into leaves of the same collection. Furthermore, each leaf intersects M in a single point which determines the asymptotic behavior of all points of that leaf in either forward or backward time.

1. INTRODUCTION

This paper, which is a sequel to [BLZ1], is devoted to the existence of stable and unstable invariant foliations for a semiflow in a Banach space. We consider a C^1 semiflow defined on a Banach space X; that is, it is continuous on $[0, \infty) \times X$, and for each $t \geq 0, T^t : X \to X$ is C^1, and

$$T^t \circ T^s(x) = T^{t+s}(x)$$

for all $t, s \geq 0$ and $x \in X$. A typical example is the solution operator for a differential equation. In [BLZ1] we proved that a compact normally hyperbolic invariant manifold M persists under small C^1 perturbations in the semiflow. We also showed that in an ϵ neighborhood of M, there exist a center-stable manifold $W^{cs}(\epsilon)$ and a center-unstable manifold $W^{cu}(\epsilon)$ which intersect in the manifold M. The purpose of this paper is to show that the center-stable manifold $W^{cs}(\epsilon)$ may be decomposed into an invariant foliation with stable leaves, while the center-unstable manifold $W^{cu}(\epsilon)$ may be decomposed into an invariant foliation with unstable leaves. These fibers or leaves give qualitative properties of the semiflow near M.

As an example, let us consider a linear system in $\mathbb{R}^{m+n}$

$$\dot{u} = Au, \qquad u \in \mathbb{R}^m,$$

$$\dot{v} = Bv, \qquad v \in \mathbb{R}^n,$$

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with $\operatorname{Re}\sigma(A) > \gamma > \operatorname{Re}\sigma(B)$, where A and B are matrices, $\sigma(A)$ and $\sigma(B)$ are the spectra of A and B, with Re denoting the real parts and $\gamma \in \mathbb{R}$ is a constant. For a given $u_0 \in \mathbb{R}^m$, after $t > 0$ the n-dimensional submanifold

$$W_0 = \Big\{(u, v) : u = u_0, v \in \mathbb{R}^n\Big\}$$

is carried by the flow to a new submanifold

$$W_t = \Big\{(u, v) : u = e^{At}u_0, v \in \mathbb{R}^n\Big\}.$$

Moreover, if both (u_1, v_1) and (u_2, v_2) lie in W_0 (i.e. $u_1 = u_2 = u_0$), then

$$||(e^{At}u_1, e^{Bt}v_1) - (e^{At}u_2, e^{Bt}v_2)|| = O(e^{\gamma t}), \quad \text{as } t \to +\infty,$$

while a trajectory starting at a point not in W_0 departs from a trajectory starting in W_0 more rapidly than $Ce^{\gamma t}$, as $t \to +\infty$. Thus, we are able to group points in $\mathbb{R}^{m+n}$ into equivalence classes according to their asymptotic behavior as $t \to +\infty$, and in this case each asymptotic class is a submanifold $u =$ constant. These submanifolds form a pseudo-stable foliation of $\mathbb{R}^{m+n}$. In general, the foliation will not consist of affine subspaces but can be thought of as perturbations of this case.

Invariant foliations with invariant manifolds have become a fundamental tool to study the qualitative properties of a flow or semiflow nearby invariant sets. They are extremely useful in that they can be used to track the asymptotic behavior of solutions and to provide coordinates in which systems of differential equations may be decoupled and normal forms derived.

In the study of the dynamics near an equilibrium or a periodic orbit, invariant manifolds and foliations serve as a convenient setting to describe the qualitative behavior of the local flows. In many cases they are useful tools for technical estimates which facilitate the study of the local bifurcation diagram (see, for example, [CLi]). Several other important concepts in dynamical systems are closely related to invariant manifolds and foliations. In finite dimensional space, the relations among invariant manifolds, invariant foliations, the λ-lemma, linearization and homoclinic bifurcation have been studied in [D1]. In [BDL], invariant foliations are used to produce smooth conjugacy of flows on different center manifolds. Kirchgraber and Palmer [KP] have recently given detailed results on invariant foliations and their applications to C^0 linearizations for finite dimensional systems. Very recently, de la Llave [Ll] studied nonresonant invariant foliations for diffeomorphisms.

In [HPS] and [F1], [F2] and [F3], invariant foliations of the stable and unstable manifolds of a normally hyperbolic invariant manifold were obtained and some of their uses demonstrated. Since then, the applications of this theory to problems from science and engineering have flourished, especially, applications to singular perturbation problems, (see, for example, [D2], [G], [GS], [HW], [Jo], [JK], [KW], [Li], [Sz] and [Te]). Recently, Jones [Jo] has given a clear discussion of the use of Fenichel's theorems as they apply to singular perturbation problems. He includes proofs of these theorems and important extensions of the λ-lemma (see also [JK]).

In the infinite dimensional setting, Li, McLaughlin, Shatah and Wiggins [LMSW] obtained invariant foliations of center-stable and center-unstable manifolds of perturbations of a circle of stationary solutions for a nonlinear Schrödinger equation. These foliations were used in tracking trajectories and completing a shooting argument to discover homoclinic orbits. In [LW], Li and Wiggins obtained invariant foliations of overflowing manifolds for a C^r ($r \geq 2$) group S^t in a Hilbert space. They did this by using the method of Hadamard's graph transform. They also applied these results to the nonlinear Schrödinger equation to recover the results of [LMSW].

Ruelle [Ru] proved a result giving invariant stable and unstable fibrations almost everywhere on a compact invariant set for a semiflow in a Hilbert space. It was assumed that the linearized time-t map is compact and injective with dense range. The results are therefore applicable to some parabolic PDE's. Mañé [Mn] extended Ruelle's results to semiflows in Banach space.

Considering the semiflow generated by a parabolic equation, Lu [Lu1] constructed infinitely many invariant manifolds as perturbations of eigenspaces of the operator obtained by linearizing at an equilibrium point. With these and corresponding invariant foliations, a new coordinate system was constructed in a neighborhood of the equilibrium point. This facilitated a proof of a Hartman-Grobman theorem for scalar parabolic PDE's, which yields that the flow near a hyperbolic equilibrium point is structurally stable. In [BL] a more general theorem on the existence of invariant foliations was proved. This theorem was then used to obtain a Hartman-Grobman result for both the phase-field system and the Cahn-Hilliard equation. In [Lu2], a Hartman-Grobman theorem for periodic orbits of time-periodic scalar parabolic PDE's is obtained by using invariant manifolds, invariant foliations and the Floquet theory obtained in [CLM]. The previously mentioned papers are concerned with dynamics in a neighborhood of an equilibrium or periodic orbit but Chow, Lin and Lu [CLL] proved a general result giving a stable fiber at each point of an inertial manifold, thereby giving a more global invariant foliation of infinite dimensional space.

Recently, Aulbach and Garay [AG] used invariant foliations to study partial linearization for noninvertible mappings near fixed points. Chen, Hale and Tan [CHT] considered a C^1 semiflow, in a Banach space X, with the linear part having an invariant splitting of X into pseudo-stable and pseudo-unstable invariant subspaces. This allowed them to prove the existence of a C^1 pseudo-unstable invariant manifold with a C^1 invariant foliation of X based on the manifold, thus giving some extension of the results in [CLL].

In this paper we start with a normally hyperbolic compact invariant manifold, M, for a semiflow and construct invariant foliations of the stable and the unstable manifold of M. What makes our analysis difficult is that we not only lack an inverse for the time-t map but we also lack a Cartesian coordinate system around our invariant manifold since the normal bundle is not assumed to be trivial.

We organize this paper as follows: in Section 2 we state our main results; in Section 3 we collect some basic lemmas from [BLZ1]; in Section 4, we construct Lipschitz stable foliations and study their basic properties; in Section 5 we prove the smoothness of the stable foliations; in Section 6, we describe how, with appropriate modifications to our methods, the results of the previous sections carry over to unstable foliations.

2. Main results

Let X be a Banach space with norm $|\cdot|$ and let T^t be a C^1 semiflow defined on X; that is, it is continuous on $[0,\infty)\times X$, and for each $t\geq 0, T^t : X\to X$ is C^1, and $T^t\circ T^s(x) = T^{t+s}(x)$ for all $t,s\geq 0$ and $x\in X$.

Let $M\subset X$ be a C^2 compact connected invariant manifold, i.e., $T^t(M)\subset M$ for each $t\geq 0$. M is said to be *normally hyperbolic* if the tangent bundle of X

restricted to M splits into three continuous subbundles

$$TX|_M = X^c \oplus X^u \oplus X^s,$$

where X^c is the tangent bundle of M, such that

(i) This splitting is invariant under the linearized semiflow, DT^t.
(ii) $DT^t|_{X^u}$ expands and does so to a greater degree than does $DT^t|_{X^c}$ while $DT^t|_{X^s}$ contracts and does so to a greater degree than does $DT^t|_{X^c}$.

The superscripts c, u and s stand for "center," "unstable," and "stable."

To be more specific, let X^c_m, X^u_m, and X^s_m denote the fibers of the vector bundles X^c, X^u and X^s at $m \in M$, respectively. To say that the splitting is invariant under the linearized semiflow we mean

(H1) For each $m \in M$ and $t \geq 0$, if $m_1 = T^t(m)$, then

$$DT^t(m)\big|_{X^\alpha_m} : X^\alpha_m \to X^\alpha_{m_1} \qquad \text{for } \alpha = c, u, s$$

and $DT^t(m)|_{X^u_m}$ is an isomorphism from X^u_m onto $X^u_{m_1}$.

The expanding and contracting condition (ii) means

(H2) There exists $t_0 \geq 0$ and $\lambda < 1$ such that for all $t \geq t_0$ and $m \in M$,

$$\lambda \inf\left\{\left|DT^t(m)x^u\right| : x^u \in X^u_m, |x^u| = 1\right\} > \max\left\{1, \|DT^t(m)\big|_{X^c_m}\|\right\}, \tag{2.1}$$

$$\lambda \min\left\{1, \inf\left\{\left|DT^t(m)x^c\right| : x^c \in X^c_m, |x^c| = 1\right\}\right\} > \|DT^t(m)\big|_{X^s_m}\|, \tag{2.2}$$

where $\|\cdot\|$ is the usual linear operator norm.

Thus, fibers X^c_m, X^u_m and X^s_m are distinguished by the growth and decay rates of the flow, much as one sees with an exponential trichotomy.

We define the direct sum of bundles in the usual way, e.g.,

$$X^u \oplus X^s \equiv \{(m, x^u + x^s) : x^u \in X^u_m \text{ and } x^s \in X^s_m, m \in M\}.$$

For $\epsilon > 0$ we shall use the notation

$$X^\alpha(\epsilon) \equiv \{(m, x^\alpha) : x^\alpha \in X^\alpha_m, m \in M, |x^\alpha| < \epsilon\}, \quad \alpha = c, u, \text{ and } s,$$

and

$$X^u(\epsilon) \oplus X^s(\epsilon) \equiv \{(m, x^u + x^s) : x^u \in X^u_m, x^s \in X^s_m, m \in M, |x^u| < \epsilon, |x^s| < \epsilon\},$$

the latter of which may be identified with a tubular neighborhood of M. The mapping which does this is $\Theta : X^u \oplus X^s \to X$ defined by $\Theta(m, x^u + x^s) = m + x^u + x^s$. For $m \in M$ we shall also use projections, Π^α_m, associated with the decomposition of $X = X^c_m \oplus X^u_m \oplus X^s_m$, for $\alpha = c, u$, and s. We assume

(H3) The mapping from $M \subset X \to \mathcal{L}(X)$, the continuous linear operators on X, defined by $m \to \Pi^\alpha_m$ is C^1 for $\alpha = c, u$, and s.

Remark. If $|\cdot|_{X^\alpha}$ is defined by $|(m, x)|_{X^\alpha} = |x|$ with $|\cdot|$ the norm in X, then the continuity in m of the above mapping gives the bundle X^α a so-called Finsler structure.

In a forthcoming book [BLZ2], we remove the assumption on the smoothness of the bundles; that is, (H3) can be removed and the assumption that the manifold M is C^2 can be relaxed to require only C^1. The technical argument needed for this generalization is lengthy and does little to illuminate the main result, therefore, it is not included here.

We first quote a lemma from [BLZ1] (Lemma 4.3)

Lemma 2.1. *For small $\epsilon > 0$, $\Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$ is a tubular neighborhood of M.*

We summarize some of the main results obtained in [BLZ1], which are needed in this paper, and state them as the following:

Theorem A. *Let $t_1 > t_0$. For each small $\epsilon > 0$, T^t has a unique C^1 center-stable manifold $W^{cs}(\epsilon)$ and a center-unstable manifold $W^{cu}(\epsilon)$ in the tubular neighborhood $\Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$. These have the following properties:*

(1) *$M = W^{cs}(\epsilon) \cap W^{cu}(\epsilon)$. For each $m \in M$, the tangent spaces at m are*

$$T_m W^{cs}(\epsilon) = X^s_m \oplus X^c_m \ \text{ and } \ T_m W^{cu}(\epsilon) = X^u_m \oplus X^c_m;$$

(2) *$T^t W^{cs}(\epsilon) \cap \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big) \subset W^{cs}(\epsilon)$. $T^t W^{cs}(\epsilon)$ converges to M as $t \to +\infty$, and*

$$W^{cs}(\epsilon) = \Big\{x \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big) : \ T^{kt_1}x \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big) \text{ for all } k > 0\Big\}; \tag{2.3}$$

(3) *$T^{t_1}\big(W^{cs}(\epsilon)\big) \subset W^{cs}(\epsilon)$;*

(4) *$T^{t_1} : W^{cu}(\epsilon) \cap (T^{t_1})^{-1}\big(W^{cu}(\epsilon)\big) \to W^{cu}(\epsilon)$ is a diffeomorphism. If we define T^{-t} on $W^{cu}(\epsilon)$ in this way, then $T^{-t}W^{cu}(\epsilon)$ converges to M as $t \to \infty$ and*

$$W^{cu}(\epsilon) = \Big\{x \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big) : \ \text{for all } k > 0, \ \text{ there exists } y_k \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big) \ \text{ satisfying } \ T^{kt_1}y_k = x\Big\}. \tag{2.4}$$

Remark. (2) implies that $W^{cs}(\epsilon)$ consists of all points in the tubular neighborhood whose forward orbits stay in the tubular neighborhood and approach M and (4) implies that $W^{cu}(\epsilon)$ consists of all points in the tubular neighborhood whose backward orbits exist and stay in the tubular neighborhood and approach M, where the backward orbits are defined by the preimages.

The proof of the existence of the two invariant manifolds is different. To obtain W^{cu}, we apply T^t to Lipschitz graphs over the bundle X^u and, essentially, find it is a contraction on the space of Lipschitz graphs. To obtain W^{cs}, the natural inclination is to reverse time and apply the preceeding argument. This is not possible, however, since we only have a semiflow. Normal hyperbolicity allows us to overcome this difficulty and we can show that for any Lipschitz graph, g, over X^s, there is a Lipschitz graph $\tilde{g}$ such that T^t maps $\tilde{g}$ **into** g. This mapping, which assigns $\tilde{g}$ to any given g, is a contraction whose fixed point is W^{cs}.

Note that $W^{cs}(\epsilon)$ and $W^{cu}(\epsilon)$ are invariant manifolds for the map T^{t_1} and also for the semiflow T^t in some sense. The properties above give some information about the asymptotic behavior of the semiflow T^t in the tubular neighborhood of M. The behavior on $W^{cs}(\epsilon)$ and $W^{cu}(\epsilon)$ is particularly important. On $W^{cs}(\epsilon)$, the motion is basically along two directions: The center direction and the stable direction. Along the stable direction the semiflow just decays exponentially. Thus, the motion along the center direction determines the asymptotic behavior. For each $x \in W^{cs}(\epsilon)$, we shall find a unique $m \in M$ so that $T^t(x)$ and $T^t(m)$ have the same asymptotic behavior at an exponential rate. In this way, $W^{cs}(\epsilon)$ can be decomposed as the disjoint union of submanifolds according to the asymptotic behavior; these manifolds being indexed by points of M. Similar statements hold for $W^{cu}(\epsilon)$. The following are the main results of this paper:

Theorem 2.2. *For small $\epsilon > 0$ there exists a unique family of C^1 submanifolds $\{W^{ss}_m(\epsilon) \ : \ m \in M\}$ of $W^{cs}(\epsilon)$ satisfying:*

(1) *For each $m \in M, M \cap W^{ss}_m(\epsilon) = \{m\}$, the tangent space $T_m W^{ss}_m(\epsilon) = X^s_m$ and $W^{ss}_m(\epsilon)$ varies continuously with respect to m on M.*
(2) *If $m_1, m_2 \in M$, $m_1 \neq m_2$, then $W^{ss}_{m_1}(\epsilon) \cap W^{ss}_{m_2}(\epsilon) = \emptyset$ and $W^{cs}(\epsilon) = \bigcup_{m\in M} W^{ss}_m(\epsilon)$.*
(3) *For $m \in M$, $T^{t_1}(W^{ss}_m(\epsilon)) \subset W^{ss}_{T^{t_1}m}(\epsilon)$.*
(4) *For all $m \in M$ and $t > 0$, $T^t(W^{ss}_m(\epsilon)) \cap \Theta(X^u(\epsilon) \oplus X^s(\epsilon)) \subset W^{ss}_{T^t(m)}(\epsilon)$.*
(5) *For $x \in W^{ss}_m(\epsilon)$ and $m \neq m_1 \in M$, we have $\frac{|T^t(x)-T^t(m)|}{|T^t(x)-T^t(m_1)|} \to 0$ exponentially as $t \to +\infty$.*
(6) *For $x, y \in W^{ss}_m(\epsilon)$, $|T^t(x) - T^t(y)| \to 0$ exponentially as $t \to +\infty$.*

For the invariant foliation of W^{cu} into unstable fibers, we have

Theorem 2.3. *For small $\epsilon > 0$ there exists a family of C^1 submanifolds $\{W^{uu}_m(\epsilon) \ : \ m \in M\}$ of $W^{cu}(\epsilon)$ satisfying:*

(1) *For each $m \in M, M \cap W^{uu}_m(\epsilon) = \{m\}$, $T_m W^{uu}_m(\epsilon) = X^u_m$ and $W^{uu}_m(\epsilon)$ varies continuously with respect to m on M.*
(2) *If $m_1, m_2 \in M$, $m_1 \neq m_2$, then $W^{uu}_{m_1}(\epsilon) \cap W^{uu}_{m_2}(\epsilon) = \emptyset$ and $W^{cu}(\epsilon) = \bigcup_{m\in M} W^{uu}_m(\epsilon)$.*
(3) *For all $m \in M$, $T^{t_1} : W^{uu}_m(\epsilon) \cap (T^{t_1})^{-1}(W^{uu}_{T^{t_1}(m)}(\epsilon)) \to W^{uu}_{T^{t_1}(m)}(\epsilon)$ is a diffeomorphism.*
(4) *For $x \in W^{uu}_m(\epsilon)$, if $T^t(x) \in \Theta(X^u(\epsilon) \oplus X^s(\epsilon))$ for all $0 < t < t_2$, for some t_2, then $T^t(x) \in W^{uu}_{T^t(m)}(\epsilon)$ for all $0 < t < t_2$.*
(5) *For $x \in W^{uu}_m(\epsilon)$ and $m \neq m_1 \in M$, we have $\frac{|T^{-t}(x)-T^{-t}(m)|}{|T^{-t}(x)-T^{-t}(m_1)|} \to 0$ exponentially as $t \to +\infty$.*
(6) *For $x, y \in W^{uu}_m(\epsilon)$, we have $|T^{-t}(x) - T^{-t}(y)| \to 0$ exponentially as $t \to +\infty$.*

We shall call each submanifold $W^{ss}_m(\epsilon)$ a stable fiber and $W^{uu}_m(\epsilon)$ an unstable fiber. The stable foliation is the decomposition of $W^{cs}(\epsilon)$ into the disjoint union of the stable fibers, and similarly the unstable foliation is the decomposition of $W^{cu}(\epsilon)$ into the disjoint union of the unstable fibers.

3. Basic lemmas

In this section we review some basic lemmas including a lemma on cone invariance and results relating three coordinate systems which were introduced in [BLZ1].

The following lemma summarizes Lemmas 4.1, 4.2, and 4.3 from [BLZ1], which give fundamental properties of the vector bundles X^α for $\alpha = u, s, c$.

Lemma 3.1. *The following statements hold:*

(i) *For $m_1, m_2 \in M$, $X^\alpha_{m_1}$ is isomorphic to $X^\alpha_{m_2}$ for $\alpha = u, s, c$;*
(ii) *X^α for $\alpha = u, s, c$, is a C^1 bundle;*
(iii) *There exists $\epsilon > 0$ such that Θ is a C^1 diffeomorphism from $X^u(\epsilon) \oplus X^s(\epsilon)$ onto a neighborhood of M.*

We recall the concept of an η-neighborhood of a point $m_0 \in M$.

Definition. For $m_0 \in M$ and $\eta \in (0, 1)$, a neighborhood U of m_0 in X is said to be an η-neighborhood if $\|\Pi^\alpha_{m_1} - \Pi^\alpha_{m_2}\| \leq \eta$ for all $m_1, m_2 \in U \cap M$, for $\alpha = c, u$, and s.

Let $\eta < \sqrt{2}-1$ and $\bar{m} \in V_m \cap M$, where V_m is an η-neighborhood of m, then we have the following estimates for the projections

$$|\Pi^\alpha_{\bar{m}} x| \geq (1-\eta)|x|, \tag{3.1}$$

for $x \in X^\alpha_m$ and

$$\|\Pi^\alpha_{\bar{m}}\,|_{X^\alpha_m}\| \leq 1+\eta \quad \text{and} \quad \|\big(\Pi^\alpha_{\bar{m}}\,|_{X^\alpha_m}\big)^{-1}\| \leq \frac{1}{1-\eta}. \tag{3.2}$$

Let $\epsilon_1 > 0$ be sufficiently small. For $m_0 \in M$, let $B(m_0, r)$ be the ball centered at m_0 of radius r, where r is chosen so that this is an η-neighborhood with $\eta < \sqrt{2}-1$. In fact, by compactness of M, r can be chosen to be independent of m_0. Next we recall three coordinate systems introduced in [BLZ1]. We express each point in a tubular neighborhood in terms of three coordinate systems as follows. For each point $x \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$, Lemma 3.1 (iii) gives the first coordinates $(m, x^u + x^s) \in X^u(\epsilon_1) \oplus X^s(\epsilon_1)$, where $x = m + x^u + x^s$. The advantage of this coordinate system is that it is global in a neighborhood of M, but the disadvantage is that the coordinate spaces vary with the base point m. For $m_0 \in M$ such that $m \in B(m_0, r)$, since $X = X^u_{m_0} \oplus X^s_{m_0} \oplus X^c_{m_0}$, there is a unique $\bar{x}^u + \bar{x}^s + \bar{x}^c \in X^u_{m_0} \oplus X^s_{m_0} \oplus X^c_{m_0}$ such that

$$m + x^u + x^s = m_0 + \bar{x}^u + \bar{x}^s + \bar{x}^c,$$

giving a Cartesian coordinate system in a neighborhood of m_0. The advantage of this system is that the coordinate spaces do not depend on the base point m, which plays a crucial role in proving the existence and smoothness of invariant foliations. If we trivialize the bundle near m_0, we have the third coordinate system modified from the first one. From (3.2) we find there exists a unique $\tilde{x}^u + \tilde{x}^s \in X^u_{m_0} \oplus X^s_{m_0}$ such that

$$m + x^u + x^s = m + \Pi^u_m \tilde{x}^u + \Pi^s_m \tilde{x}^s.$$

This defines the third coordinate system $(m, \tilde{x}^u, \tilde{x}^s)$.

We now consider the relationships between these coordinate systems. We still denote the ball with center 0 and radius ϵ in X^α_m by $X^\alpha_m(\epsilon)$. Take two points $(m_i, x^u_i + x^s_i) \in X^u_{m_i}(\epsilon_1) \oplus X^s_{m_i}(\epsilon_1)$ with $m_i \in B(m_0, r) \cap M$, $i = 1, 2$. Then we have the three representations

$$\begin{aligned} & m_i + x^u_i + x^s_i \\ & = m_0 + \bar{x}^u_i + \bar{x}^s_i + \bar{x}^c_i \\ & = m_i + \Pi^u_{m_i} \tilde{x}^u_i + \Pi^s_{m_i} \tilde{x}^s, \end{aligned} \tag{3.3}$$

where $\bar{x}^\alpha_i \in X^\alpha_{m_0}$, $\alpha = u, s, c$, $\tilde{x}^\alpha_i \in X^\alpha_{m_0}$, $\alpha = u, s$.

The next result, which is Lemma 4.5 in [BLZ1], estimates how the coordinates differ from each other.

Lemma 3.2.

$$\begin{aligned} & |\bar{x}^c_1 - \bar{x}^c_2 - (m_1 - m_2)| \\ & \leq C\epsilon_1 |m_1 - m_2| + C|m_i - m_0| \left(|\tilde{x}^u_1 - \tilde{x}^u_2| + |\tilde{x}^s_1 - \tilde{x}^s_2|\right) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} & |\bar{x}^\alpha_1 - \bar{x}^\alpha_2 - (\tilde{x}^\alpha_1 - \tilde{x}^\alpha_2)| \\ & \leq C\epsilon_1 |m_1 - m_2| + C|m_i - m_0| \left(|\tilde{x}^u_1 - \tilde{x}^u_2| + |\tilde{x}^s_1 - \tilde{x}^s_2|\right) \end{aligned} \tag{3.5}$$

for $i = 1, 2, \alpha = u, s$, where C is a positive constant which depends only on the projections.

In the remainder of this paper, we shall use C as a generic constant depending only on the projections and the time-t_1 map T^{t_1} for a fixed $t_1 > t_0$.

We shall confine our study to the tubular neighborhood $\Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$ of M. For $(m_0, x_0^u + x_0^s) \in X^u(\epsilon) \oplus X^s(\epsilon)$, and for $\mu > 0$, define the cones and truncated cones

$$K_u(m_0 + x_0^u + x_0^s, \mu) = \Big\{m_0 + x_0^u + x_0^s + x^u + x^s + x^c \ : \ x^\alpha \in X_{m_0}^\alpha \text{ for } \alpha = u, s, \text{ and } c, \mu|x^u| > (|x^c| + |x^s|)\Big\},$$

$$K_u(m_0 + x_0^u + x_0^s, \mu, \epsilon) = \Big\{m_0 + x_0^u + x_0^s + x^u + x^s + x^c \in K_u(m_0 + x_0^u + x_0^s, \mu) : |x^\alpha| < \epsilon \text{ for } \alpha = u, s, \text{ and } c\Big\},$$

$$K_{cu}(m_0 + x_0^u + x_0^s, \mu) = \Big\{m_0 + x_0^u + x_0^s + x^u + x^s + x^c \ : \ x^\alpha \in X_{m_0}^\alpha \text{ for } \alpha = u, s, \text{ and } c, \mu(|x^u| + |x^c|) > |x^s|\Big\},$$

$$K_{cu}(m_0 + x_0^u + x_0^s, \mu, \epsilon) = \Big\{m_0 + x_0^u + x_0^s + x^u + x^s + x^c \in K_{cu}(m_0 + x_0^u + x_0^s, \mu) : |x^\alpha| < \epsilon \text{ for } \alpha = u, s, \text{ and } c\Big\}.$$

Then we have the following cone invariance, which follows from Lemma 5.5 in [BLZ1].

Lemma 3.3. *For any $\lambda_1 \in (\lambda, 1)$ and $\mu > 0$, there exists ϵ^* such that if $\epsilon < \epsilon^*$, then for any $x \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$ with $T^{t_1}(x) \in \Theta(X^u(\epsilon) \oplus X^s(\epsilon))$ we have*

$$T^{t_1} K_u(x, \mu, \epsilon) \subset K_u(T^{t_1}(x), \lambda_1 \mu)$$

and

$$T^{t_1} K_{cu}(x, \mu, \epsilon) \subset K_{cu}(T^{t_1}(x), \lambda_1 \mu).$$

The compactness of M allows us to establish the following, found as Lemma 2.1 in [BLZ1].

Lemma 3.4. *For any $t > 0$, T^t is a diffeomorphism from M to M.*

The next lemma follows from the normal hyperbolicity condition. It appears as Lemma 8.2 in [BLZ1].

Lemma 3.5. *There exists $\epsilon^* > 0$ such that for $\epsilon < \epsilon^*$ if $x = m + x^u + x^s \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$, then the following estimates hold:*

(i) $\| \big(\Pi_{\tilde{m}}^u DT^{t_1}(x)|_{X_m^u}\big)^{-1} \| < \lambda$,
(ii) $\| \big(\Pi_{\tilde{m}}^u DT^{t_1}(x)|_{X_m^u}\big)^{-1} \| \|\Pi_{\tilde{m}}^c DT^{t_1}(x)|_{X_m^c}\| < \lambda$,

(iii) $\| \left(\Pi^c_{\tilde{m}} DT^{t_1}(x)|_{X^c_m}\right)^{-1} \| \|\Pi^s_{\tilde{m}} DT^{t_1}(x)|_{X^s_m}\| < \lambda,$
where $\tilde{m}$ *is given by* $T^{t_1}(x) = \tilde{m} + \tilde{x}^u + \tilde{x}^s$.

Let $\epsilon_0 \in (0, r/2)$ be fixed such that $\Theta\big(X^u(\epsilon_0) \oplus X^s(\epsilon_0)\big)$ is a tubular neighborhood of M. From Lemma 5.4 in [BLZ1] we have

Lemma 3.6. *For* $\epsilon_1 \in (0, \epsilon_0)$, *there exists a positive constant* ϵ^* *such that if* $\epsilon < \epsilon^*$, *then for all* $(m, x^u + x^s) \in \overline{X^u(\epsilon)} \oplus \overline{X^s(\epsilon)}$ *(the bar means closure) and* $\hat{x}^\alpha \in \overline{X^\alpha_m(3\epsilon)}, \alpha = u, s, c,$

$$m + x^u + x^s + \hat{x}^u + \hat{x}^s + \hat{x}^c \in \Theta\big(X^u(\epsilon_1) \oplus X^s(\epsilon_1)\big)$$

and

$$T^{t_1}(m + x^u + x^s + \hat{x}^u + \hat{x}^s + \hat{x}^c) \in \Theta\big(X^u(\epsilon_1) \oplus X^s(\epsilon_1)\big).$$

4. Existence of Lipschitz invariant stable foliation

In this section, we establish the existence of an invariant foliation (we call it a stable foliation) of $W^{cs}(\epsilon)$ for the time-t_1 map T^{t_1}. Later, we prove that this stable foliation is invariant for the semiflow T^t.

We first define a stable fiber through $x = m + x^u + x^s \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$. Consider a map associated with x

$$f_x : X^s_m(3\epsilon) \to X^u_m(\epsilon) \oplus X^c_m(\epsilon).$$

Here, we choose $\epsilon > 0$ small enough so that $\Theta\big(X^u(4\epsilon) \oplus X^s(4\epsilon)\big)$ is a tubular neighborhood of M.

Definition. $f_x : X^s_m(3\epsilon) \to X^u_m(\epsilon) \oplus X^c_m(\epsilon)$ is called a stable fiber at x of Lipschitz constant μ if

(i) $f_x(0) = 0$,
(ii) $f_x(\cdot)$ is Lipschitz continuous with Lipschitz constant μ; that is,

$$|f(y^s_1) - f(y^s_2)| \leq \mu |y^s_1 - y^s_2| \ \text{ for } \ y^s_1,\, y^s_2 \in X^s_m(3\epsilon),$$

where the norm in the direct sum of spaces is defined to be the sum of the norms. For brevity, we just call f_x a stable fiber.

We denote the graph of f_x by

$$\text{graph}(f_x) = \Big\{x + y^s + f_x(y^s) : \ y^s \in X^s_m(3\epsilon)\Big\}.$$

In the remainder of this paper, we shall fix $\mu \in (0, \frac{1}{3})$.

Next, we define a space of families of stable fibers in which we shall construct an invariant foliation of $W^{cs}(\epsilon)$. Define

$$\Gamma^{ss}_\epsilon = \Big\{F = \{f_x\}_{x \in \overline{W^{cs}(\epsilon)}} : f_x \text{ is a stable fiber and } f_x(y^s) \text{ is uniformly continuous in } x \in \overline{W^{cs}(\epsilon)}\Big\}.$$

We shall call f_x the fiber (or leaf) of F at x. The uniform continuity of the fibers f_x in x will be defined precisely, later.

The main theorem in this section is

Theorem 4.1. *There exists $\epsilon^* > 0$ such that for $\epsilon < \epsilon^*, W^{cs}(\epsilon)$ has a unique invariant foliation given by*

$$W^{cs}(\epsilon) = \bigcup_{m\in M} W^{ss}_m(\epsilon)$$

where $W^{ss}_m(\epsilon) =$ graph$(f_m) \cap W^{cs}(\epsilon)$ satisfying:

(1) *f_m is a stable fiber.*
(2) *$W^{ss}_m(\epsilon)$ is continuous in $m \in M$.*
(3) *$W^{ss}_{m_1}(\epsilon) \cap W^{ss}_{m_2}(\epsilon) = \varnothing$ for $m_1, m_2 \in M$, $m_1 \neq m_2$.*
(4) *For $m \in M$, $T^{t_1}\big(W^{ss}_m(\epsilon)\big) \subset W^{ss}_{T^{t_1}(m)}(\epsilon)$.*
(5) *For all $m \in M$ and $t > 0$, $T^t\big(W^{ss}_m(\epsilon)\big) \cap \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big) \subset W^{ss}_{T^t(m)}(\epsilon)$.*
(6) *For $x \in W^{ss}_m(\epsilon)$ and $m \neq m_1 \in M$, we have $\frac{|T^t(x)-T^t(m)|}{|T^t(x)-T^t(m_1)|} \to 0$ exponentially as $t \to +\infty$.*
(7) *For $x, y \in W^{ss}_m(\epsilon)$, $|T^t(x) - T^t(y)| \to 0$ exponentially as $t \to +\infty$.*

The proof of this theorem is based on the following propositions and lemmas.

Proposition 4.2. *There exists $\epsilon^* > 0$ such that for $0 < \epsilon < \epsilon^*$ and any $F \in \Gamma^{ss}_\epsilon$, there exists a unique $\tilde{F} \in \Gamma^{ss}_\epsilon$ such that each fiber of $\tilde{F}$ is mapped into a fiber of F by T^{t_1}; that is, given any $x_1 = m_1 + x^u_1 + x^s_1 \in W^{cs}(\epsilon)$ and any $y^s_1 \in X^s_{m_1}(3\epsilon)$, there exists a unique $y^s_2 \in X^s_{m_2}(3\epsilon)$, such that*

$$T^{t_1}\big(x_1 + y^s_1 + \tilde{f}_{x_1}(y^s_1)\big) = x_2 + y^s_2 + f_{x_2}(y^s_2), \tag{4.1}$$

where m_2 and x_2 are given by $T^{t_1}(x_1) = x_2 = m_2 + x^u_2 + x^s_2$.

This will be proved through a sequence of lemmas.

Throughout this section, for $x_1 = m_1 + x^u_1 + x^s_1 \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$, we denote $T^{t_1}(x_1)$ by x_2 and write $x_2 = T^{t_1}(x_1) = m_2 + x^u_2 + x^s_2$.

Lemma 4.3. *There exists $\epsilon^* > 0$ such that if $0 < \epsilon < \epsilon^*$, then for all $x_1 \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$ with $T^{t_1}(x_1) \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$, $y^s_1 \in X^s_{m_1}(3\epsilon)$ and stable fiber f_{x_2}, there exists a unique $y^u_1 + y^c_1 \in X^u_{m_1}(\epsilon) \oplus X^c_{m_1}(\epsilon)$ such that*

$$T^{t_1}(x_1 + y^s_1 + y^u_1 + y^c_1) \in \text{graph}(f_{x_2}). \tag{4.2}$$

Proof. For $x_1 = m_1 + x^u_1 + x^s_1 \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$ with $T^{t_1}(x_1) \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$, we write $x_2 = T^{t_1}(x_1)$ as

$$x_2 = T^{t_1}(x_1) = m_2 + x^u_2 + x^s_2,$$

where $x^\alpha_2 \in X^\alpha_{m_2}(\epsilon)$ for $\alpha = u, s$. For any $y^u + y^c \in \overline{X^u_{m_1}(\epsilon)} \oplus \overline{X^c_m(\epsilon)}$, we write

$$T^t(x_1 + y^u + y^s_1 + y^c) = x_2 + \bar{y}^u + \bar{y}^s + \bar{y}^c \tag{4.3}$$

where $\bar{y}^\alpha \in X^\alpha_{m_2}$ for $\alpha = u, s, c$.

It is enough to prove that there is a unique $y^u + y^c \in X^u_{m_1}(\epsilon) \oplus X^c_{m_1}(\epsilon)$ such that $\bar{y}^u + \bar{y}^c = f_{x_2}(\bar{y}^s)$. To find such $y^u + y^c$, we consider a map $A = A^u + A^c$ from $X^u_{m_1}(\epsilon) \oplus X^c_{m_1}(\epsilon)$ to $X^u_{m_1} \oplus X^c_{m_1}$ defined by

$$A^u(y^u + y^c) = \Big(\Pi^u_{m_2} DT^{t_1}(m_1)|_{X^u_{m_1}}\Big)^{-1}(\tilde{y}^u - \bar{y}^u) + y^u$$

and

$$A^c(y^u+y^c)=\left(\Pi^c_{m_2}DT^{t_1}(m_1)|_{X^c_{m_1}}\right)^{-1}(\tilde{y}^c-\bar{y}^c)+y^c,$$

where $\tilde{y}^u+\tilde{y}^c=f_{x_2}(\bar{y}^s)$.

We claim that A is a contraction on $\overline{X^u_{m_1}(\epsilon)}\oplus\overline{X^c_{m_1}(\epsilon)}$.

We first prove that A is well defined. Note that $DT^{t_1}(m_1):X^u_{m_1}\to X^u_{\overline{m_1}}$ and $DT^{t_1}(m_1):X^c_{m_1}\to X^c_{\overline{m_1}}$ have bounded inverses from hypothesis (H2), where $\overline{m_1}=T^{t_1}(m_1)$. From Lemma 3.5, by choosing ϵ^* small enough, both $\Pi^u_{m_2}DT^{t_1}(m_1)|_{X^u_{m_1}}$ and $\Pi^c_{m_2}DT^{t_1}(m_1)|_{X^c_{m_1}}$ have bounded inverses.

Applying the projection $\Pi^s_{m_2}$ to (4.3) we have

$$\bar{y}^s=\Pi^s_{m_2}\big(T^{t_1}(x_1+y^u+y^s_1+y^c)-x_2\big).$$

Using the Taylor expansion of $T^{t_1}(x_1+y^u+y^s_1+y^c)$ and hypothesis (H2), we obtain

$$\begin{aligned}|\bar{y}^s|&=|\Pi^s_{m_2}\Big(T^{t_1}(m_1+x^u_1+x^s_1+y^u+y^s_1+y^c)-T^{t_1}(m_1+x^u_1+x^s_1)\Big)|\\&\le|\Pi^s_{m_2}DT^{t_1}(m_1)(y^u+y^s_1+y^c)|+O(\epsilon)|y^u+y^s_1+y^c|\\&\le|\Pi^s_{\bar{m}_1}DT^{t_1}(m_1)y^s_1|+O(\epsilon)|y^u+y^s_1+y^c|\\&\le\lambda|y^s_1|+O(\epsilon)|y^u+y^s_1+y^c|\\&\le\big(3\lambda+O(\epsilon)\big)\epsilon<3\epsilon\end{aligned}\tag{4.4}$$

provided that ϵ^* is sufficiently small. Here $0<\lambda<1$ is used. Hence, A is well defined.

Next we show that $A:\overline{X^u_{m_1}(\epsilon)}\oplus\overline{X^c_{m_1}(\epsilon)}\to X^u_{m_1}\oplus X^c_{m_1}$ is a contraction. For $y^u_1+y^c_1$, $y^u_2+y^c_2\in X^u_{m_1}(\epsilon)\oplus X^c_{m_1}(\epsilon)$. We write

$$T^{t_1}(x_1+y^u_i+y^s_1+y^c_i)=x_2+\bar{y}^u_i+\bar{y}^s_i+\bar{y}^c_i\tag{4.5}$$

and set

$$\tilde{y}^u_i+\tilde{y}^c_i=f_{x_2}(\bar{y}^s_i)\tag{4.6}$$

for $i=1,2$. Note that from (4.4), $|\bar{y}^s_i|<3\epsilon$.

Let $\ell(\tau)=(1-\tau)(y^u_1+y^c_1)+\tau(y^u_2+y^c_2)+x_1+y^s_1$ for $0\le\tau\le1$.

For $\alpha=u,c$, compute

$$\begin{aligned}&\left(\Pi^\alpha_{m_2}DT^{t_1}(m_1)|_{X^\alpha_{m_1}}\right)^{-1}(\bar{y}^\alpha_2-\bar{y}^\alpha_1)\\&=\left(\Pi^\alpha_{m_2}DT^{t_1}(m_1)|_{X^\alpha_{m_1}}\right)^{-1}\int_0^1\Pi^\alpha_{m_2}DT^{t_1}\big(\ell(\tau)\big)(y^u_2-y^u_1+y^c_2-y^c_1)d\tau\\&=(y^\alpha_2-y^\alpha_1)+O(\epsilon)(|y^u_2-y^u_1|+|y^c_2-y^c_1|).\end{aligned}\tag{4.7}$$

In order to show that A is a contraction, we also need to estimate

$$\begin{aligned}|\bar{y}^s_2-\bar{y}^s_1|&=|\int_0^1\Pi^s_{m_2}DT\big(\ell(\tau)\big)(y^u_2-y^u_1+y^c_2-y^c_1)d\tau|\\&\le O(\epsilon)(|y^u_2-y^u_1|+|y^c_2-y^c_1|).\end{aligned}\tag{4.8}$$

From (4.6), by using (4.8), we obtain

$$|\tilde{y}^u_2-\tilde{y}^u_1|+|\tilde{y}^c_2-\tilde{y}^c_2|\le O(\epsilon)(|y^u_2-y^u_1|+|y^c_2-y^c_1|).\tag{4.9}$$

Thus, from (4.7) and (4.9) we have

$$
\begin{aligned}
&|A(y_2^u + y_2^c) - A(y_1^u + y_1^c)| \\
&= |\Big(\Pi_{m_2}^u DT^{t_1}(m_1)|_{X_{m_1}^u}\Big)^{-1}(\tilde{y}_2^u - \tilde{y}_1^u - \bar{y}_2^u + \bar{y}_1^u) + y_2^u - y_1^u| \\
&+ |\Big(\Pi_{m_2}^c DT^{t_1}(m_1)|_{X_{m_1}^c}\Big)^{-1}(\tilde{y}_2^c - \tilde{y}_1^c - \bar{y}_2^u + \bar{y}_1^u) + y_2^c - y_1^c| \\
&\le O(\epsilon)(|y_2^u - y_1^u| + |y_2^c - y_1^c|). \qquad (4.10)
\end{aligned}
$$

Here Lemmas 3.4 and 3.5 are used to get a bound for $\|(\Pi_{m_2}^\alpha DT^{t_1}(m_1)|_{X_{m_1}^\alpha})^{-1}\|$ for $\alpha = u, c$. This shows that A is a contraction by choosing ϵ^* sufficiently small. To complete the proof, we show that A maps $\overline{X_{m_1}^u(\epsilon)} \oplus \overline{X_{m_1}^c(\epsilon)}$ into $X_{m_1}^u(\epsilon) \oplus X_{m_1}^c(\epsilon)$. We first estimate $A(0)$. Again using (4.5), we have

$$
\bar{y}^\alpha = \Pi_{m_2}^\alpha\big(T^{t_1}(x_1 + y_1^s) - T^{t_1}(x_1)\big) \qquad (4.11)
$$

for $\alpha = u, s, c$.

For $\alpha = u, c$ we estimate

$$
\begin{aligned}
|\bar{y}^\alpha| &= |\Pi_{m_2}^\alpha\big(T^{t_1}(x_1 + y_1^s) - T^{t_1}(x_1)\big)| \\
&\le O(\epsilon)|y_1^s|.
\end{aligned}
$$

We also estimate

$$
\begin{aligned}
|\bar{y}^s| &= |\Pi_{m_2}^s\big(T^{t_1}(x_1 + y_1^s) - T^{t_1}(x_1)\big)| \\
&\le \|DT^{t_1}(m_1)|_{X_{m_1}^s}\||y_1^s| + O(\epsilon)|y_1^s|.
\end{aligned}
$$

Hence, from (4.6) we obtain

$$
|\tilde{y}^u| + |\tilde{y}^c| \le \Big(\mu\|DT^{t_1}(m_1)|_{X_{m_1}^s}\| + O(\epsilon)\Big)|y_1^s|.
$$

Thus, for $\alpha = u, c$, we have

$$
\begin{aligned}
&|\Big(\Pi_{m_2}^\alpha DT^{t_1}(m_1)|_{X_{m_1}^\alpha}\Big)^{-1}(\tilde{y}^\alpha - \bar{y}^\alpha)| \\
&\le \Big(\mu\|\Big(DT^{t_1}(m_1)|_{X_{m_1}^\alpha}\Big)^{-1}\|\|DT^{t_1}(m_1)|_{X_{m_1}^s}\| + O(\epsilon)\Big)|y_1^s| \\
&\le \big(\mu\lambda + O(\epsilon)\big)|y_1^s| \\
&\le \big(3\mu\lambda + O(\epsilon)\big)\epsilon \qquad (4.12)
\end{aligned}
$$

where Lemmas 3.4 and 3.5 are used.

Hence, for $\alpha = u, c$,

$$
|A^\alpha(0)| \le \big(3\mu\lambda + O(\epsilon)\big)\epsilon.
$$

For any $y^u + y^c \in \overline{X_{m_1}^u(\epsilon)} \oplus \overline{X_{m_1}^c(\epsilon)}$, from (4.10) it follows that for $\alpha = u, c$,

$$
\begin{aligned}
|A^\alpha(y^u + y^c)| &\le |A^\alpha(y^u + y^c) - A^\alpha(0)| + |A^\alpha(0)| \\
&\le \big(3\mu\lambda + O(\epsilon)\big)\epsilon < \epsilon
\end{aligned}
$$

provided that ϵ^* is sufficiently small since $0 < \mu < \frac{1}{3}$ and $0 < \lambda < 1$. This proves the claim that A is a contraction from $\overline{X_{m_1}^u(\epsilon)} \oplus \overline{X_{m_1}^c(\epsilon)}$ into $X_{m_1}^u(\epsilon) \oplus X_{m_1}^s(\epsilon)$. By the contraction mapping theorem, A has a unique fixed point $y_1^u + y_1^c \in X_{m_1}^u(\epsilon) \oplus X_{m_1}^c(\epsilon)$, and thus, the corresponding $\bar{y}_1^u + \bar{y}_1^c + \bar{y}_1^c$ satisfies $\bar{y}_1^u + \bar{y}_1^c = f_{x_2}(\bar{y}_1^s)$; that is,

$$
T(x_1 + y_1^u + y_1^s + y_1^c) \in \text{graph}(f_{x_2}).
$$

This completes the proof of the lemma. □

Define a map $\tilde{f}_{x_1}$ from $X^s_{m_1}(3\epsilon)$ to $X^u_{m_1}(\epsilon) \oplus X^c_{m_1}(\epsilon)$ by

$$\tilde{f}_{x_1}(y^s_1) = y^u_1 + y^c_1.$$

Lemma 4.3 shows that this map is well defined.

Clearly, by Lemma 3.3, we have

Lemma 4.4. $\tilde{f}_{x_1}(0) = 0$ *and Lip* $(\tilde{f}_{x_1}) \leq \lambda_1 \mu$, *where* $0 < \lambda < \lambda_1 < 1$.

We now study the continuity of the fibers with respect to the base point x.

Let $x_n = m_n + x^u_n + x^s_n \in \Theta\big(X^u(\epsilon) \oplus X^u(\epsilon)\big)$, for $n = 0, 1, 2, \ldots$, and suppose $x_n \to x_0$ as $n \to \infty$. Consider a sequence of stable fibers

$$f_{x_n} : X^s_{m_n}(3\epsilon) \to X^u_{m_n}(\epsilon) \oplus X^c_{m_n}(\epsilon).$$

Definition. f_{x_n} is said to be convergent to f_{x_0} as $n \to \infty$ if for any $y^s_n \in X^s_{m_n}(3\epsilon)$ satisfying

$$x_n + y^s_n \to x_0 + y^s_0,$$

then $x_n + y^s_n + f_{x_n}(y^s_n) \to x_0 + y^s_0 + f_{x_0}(y^s_0)$ as $n \to \infty$.

We denote this by

$$f_{x_n} \to f_{x_0} \text{ as } n \to \infty.$$

Definition. f_{x_n} is said to be uniformly convergent to f_{x_0} if for any $\kappa > 0$, there exists an integer $N > 0$ such that for all $n > N$, if $y^s_0 \in X^s_{m_0}(3\epsilon)$ and $y^s_n = (\Pi^s_{m_0}|_{X^s_{m_n}})^{-1} y^s_0 \in X^s_{m_n}(3\epsilon)$, then

$$|x_n + y^s_n + f_{x_n}(y^s_n) - \big(x_0 + y^s_0 + f_{x_0}(y^s_0)\big)| < \kappa.$$

Remark. The above definition of convergence is equivalent to saying that for any sequence $y^s_n \in X^s_{m_n}(3\epsilon)$ satisfying $|\bar{y}^s_n - y^s_0| \to 0$, where $\bar{y}^s_n = \Pi^s_{m_0} y^s_n$, then

$$(\Pi^u_{m_0} \oplus \Pi^c_{m_0}) f_{x_n}(y^s_n) \to f_{x_0}(y^s_0).$$

Let $x_n = m_n + x^u_n + x^s_n \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$ with $T^{t_1}(x_n) \in \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big)$ for $n = 0, 1, 2, \ldots$, and $x_n \to x_0$. Denote $\tilde{x}_n = T^{t_1}(x_n) = \tilde{m}_n + \tilde{x}^u_n + \tilde{x}^s_n$. Consider a sequence of stable fibers $f_{\tilde{x}_n}$ with base point $\tilde{x}_n$. By Lemma 4.3, for each n, there exists a unique stable fiber $\tilde{f}_{x_n}$ such that

$$T^{t_1}\big(x_n + y^s_n + \tilde{f}_{x_n}(y^s_n)\big) \in \text{ graph } (f_{\tilde{x}_n}).$$

Note that $\tilde{x}_n \to \tilde{x}_0$ as $n \to \infty$. Then, we have

Lemma 4.5. (i) *If* $f_{\tilde{x}_n}$ *converges to* $f_{\tilde{x}_0}$, *then* $\tilde{f}_{x_n}$ *converges to* $\tilde{f}_{x_0}$;

(ii) *If* $f_{\tilde{x}_n}$ *converges uniformly to* $f_{\tilde{x}_0}$, *then* $\tilde{f}_{x_n}$ *converges uniformly to* $\tilde{f}_{x_0}$.

Proof. Let $y^s_n \in X^s_{m_n}(3\epsilon)$. Suppose that $x_n + y^s_n \to x_0 + y^s_0$ as $n \to \infty$. We shall show

$$x_n + y^s_n + \tilde{f}_{x_n}(y^s_n) \to x_0 + y^s_0 + \tilde{f}_{x_0}(y^s_0).$$

For each n, by Lemma 4.3, we write

$$T^{t_1}\big(x_n + y^s_n + \tilde{f}_{x_n}(y^s_n)\big) = \tilde{x}_n + \tilde{y}^u_n + \tilde{y}^s_n + \tilde{y}^c_n. \tag{4.13}$$

Note that $\tilde{y}^u_n + \tilde{y}^c_n = f_{\tilde{x}_n}(\tilde{y}^s_n)$.

Let $\tilde{f}_{x_n}(y^s_n) = y^u_n + y^c_n$. For $\tau \in [0, 1]$, we denote

$$\ell_n(\tau) = \tau(x_n + y^u_n + y^s_n + y^c_n) + (1 - \tau)(x_0 + y^u_0 + y^s_0 + y^c_0). \tag{4.14}$$

A simple computation gives

$$|\ell_n(\tau) - m_0| \le C\epsilon, |T^{t_1}(m_0) - \tilde{m}_0| \le C\epsilon,$$

where C is a positive constant. Thus, from (4.13), we obtain for $\alpha = u, s, c$,

$$\begin{aligned}
&\Pi^\alpha_{\tilde{m}_0}(\tilde{y}^u_n + \tilde{y}^s_n + \tilde{y}^c_n) - \tilde{y}^\alpha_0 \\
&= \Pi^\alpha_{\tilde{m}_0}\Big(\tilde{x}_n + \tilde{y}^u_n + \tilde{y}^s_n + \tilde{y}^c_n - (\tilde{x}_0 + \tilde{y}^u_0 + \tilde{y}^s_0 + \tilde{y}^c_0) + \tilde{x}_0 - \tilde{x}_n)\Big) \\
&= \Pi^\alpha_{\tilde{m}_0}(\tilde{x}_0 - \tilde{x}_n) + \Pi^\alpha_{\tilde{m}_0}\Big(T^{t_1}\big(x_n + y^s_n + \tilde{f}_{x_n}(y^s_n)\big) \\
&\qquad\qquad - T^{t_1}\big(x_0 + y^s_0 + \tilde{f}_{x_0}(y^s_0)\big)\Big) \qquad (4.15)\\
&= \Pi^\alpha_{\tilde{m}_0}(\tilde{x}_0 - \tilde{x}_n) \\
&\quad + \Pi^\alpha_{\tilde{m}_0}\int_0^1 DT^{t_1}\big(\ell_n(\tau)\big)\big(x_n - x_0 + y^s_n - y^s_0 + \tilde{f}_{x_n}(y^s_n) - \tilde{f}_{x_0}(y^s_0)\big)d\tau \\
&= I_n + \big(\Pi^\alpha_{\bar{m}_0} DT^{t_1}(m_0) + O(\epsilon)\big)J_n,
\end{aligned}$$

where $\bar{m}_0 = T^{t_1}(m_0)$, I_n is a quantity which approaches zero as $n \to \infty$, and $J_n = \Pi^u_{m_0} y^u_n - y^u_0 + \Pi^c_{m_0} y^c_n - y^c_0$.

We want to show $J_n \to 0$ as $n \to \infty$.

For $\alpha \ne \alpha'$, where $\alpha, \alpha' = u, s, c$, we have

$$\begin{aligned}
|\Pi^\alpha_{\tilde{m}_0}\tilde{y}^{\alpha'}_n| &= |(\Pi^\alpha_{\tilde{m}_0} - \Pi^\alpha_{\tilde{m}_n})\tilde{y}^{\alpha'}_n| \\
&\le 3\epsilon\|\Pi^\alpha_{\tilde{m}_0} - \Pi^\alpha_{\tilde{m}_n}\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence, by (4.15) we obtain

$$\Pi^\alpha_{\tilde{m}_0}\tilde{y}^\alpha_n - \tilde{y}^\alpha_0 = \hat{I}_n + \big(\Pi^\alpha_{\bar{m}_0} DT^{t_1}(m_0) + O(\epsilon)\big)J_n, \qquad (4.16)$$

where $\hat{I}_n \to 0$ as $n \to \infty$.

Let $\alpha = s$ in (4.16) and we estimate

$$|\Pi^s_{\tilde{m}_0}\tilde{y}^s_n - \tilde{y}^s_0| \le |\hat{I}_n| + O(\epsilon)|J_n|. \qquad (4.17)$$

Again, from (4.15) it follows that

$$\begin{aligned}
&|\Pi^u_{\tilde{m}_0}\tilde{y}^u_n - \tilde{y}^u_0 + \Pi^c_{\tilde{m}_0}\tilde{y}^c_n - \tilde{y}^c_0| \\
&\ge |\Pi^u_{\bar{m}_0} DT^{t_1}(m_0)(\Pi^u_{m_0} y^u_n - y^u_0) + \Pi^c_{\bar{m}_0} DT^{t_1}(m_0)(\Pi_{m_0} y^c_n - y^c_0)| \\
&\quad - |\hat{I}_n| - O(\epsilon)|J_n| \\
&= |DT^{t_1}(m_0)(\Pi^u_{m_0} y^u_n - y^u_0) + DT^{t_1}(m_0)(\Pi^c_{m_0} y^c_n - y^c_0)| \qquad (4.18)\\
&\quad - |\hat{I}_n| - O(\epsilon)|J_n| \\
&\ge \frac{1}{C}|\Pi^u_{m_0} y^u_n - y^u_0 + \Pi^c_{m_0} y^c_n - y^c_0| - |\hat{I}_n| - O(\epsilon)|J_n| \\
&\ge \big(\frac{1}{C} - O(\epsilon)\big)|J_n| - |\hat{I}_n|,
\end{aligned}$$

where Lemma 3.4 and Lemma 3.5 are used.

Let $\bar{y}^u_n + \bar{y}^c_n = f_{\tilde{x}_n}\Big(\big(\Pi^s_{\tilde{m}_n}|_{X^s_{\tilde{m}_n}}\big)^{-1}\tilde{y}^s_0\Big)$, which is well defined for large n. Since $f_{\tilde{x}_n} \to f_{\tilde{x}_0}$, we have

$$\Pi^u_{\tilde{m}_0}\bar{y}^u_n \to \tilde{y}^u_0 \text{ and } \Pi^c_{\tilde{m}_0}\bar{y}^c_n \to \tilde{y}^c_0 \qquad (4.19)$$

as $n \to \infty$.

From (4.18), by using (4.17), we have for large n

$$\begin{aligned}
&\big(C^{-1}-O(\epsilon)\big)|J_n|-|\hat{I}_n|\\
&\le |\Pi^u_{\tilde{m}_0}(\tilde{y}^u_n-\bar{y}^u_n)+\Pi^c_{\tilde{m}_0}(\tilde{y}^c_n-\bar{y}^c_n)|\\
&\quad +|\Pi^u_{\tilde{m}_0}\bar{y}^u_n-\tilde{y}^u_0|+|\Pi^c_{\tilde{m}_0}\bar{y}^c_n-\tilde{y}^c_0|\\
&\le \mu(\|\Pi^u_{\tilde{m}_0}\|+\|\Pi^c_{\tilde{m}_0}\|)|\tilde{y}^s_n-(\Pi^s_{\tilde{m}_0}|_{X^s_{\tilde{m}_n}})^{-1}\tilde{y}^s_0|\\
&\quad +|\Pi^u_{\tilde{m}_0}\bar{y}^u_n-\tilde{y}^u_0|+|\Pi^c_{\tilde{m}_0}\bar{y}^c_n-\tilde{y}^c_0|\\
&\le C(|\hat{I}_n|+O(\epsilon)|J_n|)+|\Pi^u_{\tilde{m}_0}\bar{y}^u_n-\tilde{y}_0|+|\Pi^c_{\tilde{m}_0}\bar{y}^c_n-\tilde{y}^c_0|,
\end{aligned} \tag{4.20}$$

where C is a positive constant depending only on projections.

Therefore, if ϵ^* is sufficiently small, $J_n\to 0$ as $n\to\infty$, which then implies $f_{x_n}\to f_{x_0}$. The uniform convergence follows from the same estimate (4.20). This completes the proof of the lemma. □

This lemma is the key to showing the continuity of the stable foliation with respect to base point in M.

The next lemma is needed to establish the fact that the semiflow induces a graph transform which is a contraction on Γ^{ss}_ϵ.

We first define a norm for a stable fiber as follows

$$\|f_x\|=\sup_{\substack{y^s\in X^s_m(3\epsilon)\\ y^s\neq 0}}\frac{|f_x(y^s)|}{|y^s|}.$$

It is easy to see that Γ^{ss}_ϵ is complete under the norm

$$||F||=\sup\{||f_x||\ :\ f_x\in F\}.$$

For $x_1\in\Theta\big(X^u(\epsilon)\oplus X^s(\epsilon)\big)$ with $T^{t_1}(x_1)\in\Theta\big(X^u(\epsilon)\oplus X^s(\epsilon)\big)$, let $x_2=T^{t_1}(x_1)$. Given any two stable fibers at x_2,

$$f_{x_2},\, g_{x_2},$$

by Lemma 4.3, there exists two corresponding fibers $\tilde{f}_{x_1}$ and $\tilde{g}_{x_1}$ which satisfy (4.1). Furthermore, we have

Lemma 4.6. *There exists $\epsilon^*>0$ such that if $0<\epsilon<\epsilon^*$, then*

$$\|\tilde{f}_{x_1}-\tilde{g}_{x_1}\|\le\lambda_1\|f_{x_2}-g_{x_2}\|$$

where $\lambda<\lambda_1<1$.

Proof. For any $y^s_1\in X^s_{m_1}(3\epsilon)$, by Lemma 4.3, there exist y^s_2, $\tilde{y}^s_2$ such that

$$T^{t_1}\big(x_1+y^s_1+\tilde{f}_{x_1}(y^s_1)\big)=x_2+y^s_2+f_{x_2}(y^s_2), \tag{4.21}$$

and

$$T^{t_1}\big(x_1+y^s_1+\tilde{g}_{x_1}(y^s_1)\big)=x_2+\tilde{y}_2+g_{x_2}(\tilde{y}^s_2). \tag{4.22}$$

Let

$$\begin{aligned}
y^u_1+y^c_1&=\tilde{f}_{x_1}(y^s_1),\\
y^u_2+y^c_2&=f_{x_2}(y^s_2),\\
\tilde{y}^u_2+\tilde{y}^c_2&=g_{x_2}(\tilde{y}^s_2).
\end{aligned}$$

Then, from (4.21) and (4.22) it follows that

$$\begin{aligned}&(\tilde{y}_2^u+\tilde{y}_2^s+\tilde{y}_2^c)-(y_2^u+y_2^s+y_2^c)\\ &=DT^t(m_1)\big(\tilde{g}_{x_1}(y_1^s)-\tilde{f}_{x_1}(y_1^s)\big)+O(\epsilon)|\tilde{g}_{x_1}(y_1^s)-\tilde{f}_{x_1}(y_1^s)|.\end{aligned}\tag{4.23}$$

Applying the projection $\Pi_{m_2}^{\alpha}$ to (4.23), we obtain

$$\begin{aligned}\tilde{y}_2^{\alpha}-y_2^{\alpha}=\Pi_{m_2}^{\alpha}DT^{t_1}(m_1)\big(\tilde{g}_{x_1}(y_1^s)-\tilde{f}_{x_1}(y_1^s)\big)\\ +O(\epsilon)|\tilde{g}_{x_1}(y_1^s)-\tilde{f}_{x_1}(y_1^s)|.\end{aligned}\tag{4.24}$$

Let $\alpha=s$ in (4.24), then,

$$\tilde{y}_2^s-y_2^s=O(\epsilon)|\tilde{g}_{x_1}(y_1^s)-\tilde{f}_{x_1}(y_1^s)|.\tag{4.25}$$

On the other hand, from (4.24), by using (4.25), we obtain

$$\begin{aligned}&|g_{x_2}(y_2^s)-f_{x_2}(y_2^s)|\\ &\geq|g_{x_2}(\tilde{y}_2^s)-f_{x_2}(y_2^s)|-|g_{x_2}(\tilde{y}_2^s)-g_{x_2}(y_2^s)|\\ &\geq|\tilde{y}_2^u-y_2^u|+|\tilde{y}_2^c-y_2^c|-\mu|\tilde{y}_2^s-y_2^s|\\ &\geq|\Pi_{m_2}^uDT^{t_1}(m_1)(\tilde{y}_1^u-y_1^u)|+|\Pi_{m_2}^cDT^{t_1}(m_1)(\tilde{y}_1^c-y_1^c)|\\ &\quad-O(\epsilon)|\tilde{g}_{x_1}(y_1^s)-\tilde{f}_{x_1}(y_1^s)|.\end{aligned}\tag{4.26}$$

From (4.21) it follows that

$$y_2^s=\Pi_{m_2}^sDT^{t_1}(m_1)\big(y_1^s+\tilde{f}_{x_1}(y_1^s)\big)+O(\epsilon)\big(|y_1^s|+\tilde{f}_{x_1}(y_1^s)\big)$$

which implies

$$|y_2^s|\leq\|\Pi_{m_2}^sDT^{t_1}(m_1)|_{X_{m_1}^s}\||y_1^s|+O(\epsilon)|y_1^s|.$$

Therefore, by (4.26) together with Lemma 3.5, we obtain

$$\frac{|\tilde{g}_{x_1}(y_1^s)-\tilde{f}_{x_1}(y_1^s)|}{|y_1^s|}\leq\lambda_1\frac{|g_{x_2}(y_2^s)-f_{x_2}(y_2^s)|}{|y_2^s|},$$

provided that ϵ^* is sufficiently small. Thus,

$$\|\tilde{g}_{x_1}-\tilde{f}_{x_1}\|\leq\lambda_1\|g_{x_2}-f_{x_2}\|.$$

This completes the proof of the lemma. □

Proof of Proposition 4.2. By Lemma 4.3, for $F\in\Gamma_{\epsilon}^{ss}$, we may define a family of stable fibers

$$\tilde{F}=\left\{\tilde{f}_x:x\in\overline{W^{cu}(\epsilon)}\right\}$$

which satisfies

$$T^{t_1}(\text{graph }(\tilde{f}_x))\subset\text{ graph}(f_{T^{t_1}(x)}),$$

and so induces a graph transform

$$\mathcal{F}^{ss}(F)=\tilde{F}.$$

Clearly, such $\tilde{F}$ is unique. To complete the proof, we must show that $\tilde{F}\in\Gamma_{\epsilon}^{ss}$. It is enough to show that $\tilde{f}_x$ is uniformly continuous in x. Since f_x is uniformly continuous in x, by using Lemma 4.5, we obtain that $\tilde{f}_x$ is uniformly continuous in x. This completes the proof. □

We now have the graph transform $\mathcal{F}^{ss}$ defined on Γ_ϵ^{ss}. It follows from Lemma 4.6 that $\mathcal{F}^{ss}$ is a contraction, hence, it has a unique fixed point

$$F^{ss} = \Big\{ f_x^{ss} : x \in W^{cs}(\epsilon) \Big\}.$$

The next result shows that each fiber lies on the center-stable manifold $W^{cs}(\tilde{\epsilon})$.

Proposition 4.7. *For each $x \in W^{cs}(\epsilon)$, graph $(f_x^{ss}) \subset W^{cs}(\tilde{\epsilon})$.*

Proof. For any $x_0 \in W^{cs}(\epsilon)$, let

$$x_k = (T^{t_1})^k(x_0).$$

Note that $x_k \in W^{cs}(\epsilon)$. We write

$$x_k = m_k + x_k^u + x_k^s$$

where $x_k^\alpha \in X_{m_k}^\alpha(\epsilon)$, for $\alpha = u, s$.

Let $y_{0,i}^s \in X_{m_0}^s(3\epsilon)$, for $i = 1, 2$. From the invariance of F^{ss}, there exist $y_{k,i}^s \in X_{m_k}^s(3\epsilon)$ such that for $i = 1, 2$

$$T^{kt_1}\big(x_0 + y_{0,i}^s + f_{x_0}^{ss}(y_{0,i}^s)\big) = x_k + y_{k,i}^s + f_{x_k}^{ss}(y_{k,i}^s).$$

An elementary computation gives

$$|y_{k,2}^s - y_{k,1}^s| \le \lambda_1^k |y_{0,2}^s - y_{0,1}^s| \text{ for all } k \ge 1 \tag{4.27}$$

provided that ϵ^* is sufficiently small. Therefore, Theorem A (2) completes the proof of the proposition. □

The next lemma gives a geometric description of F^{ss} in terms of cones. For $x_0 \in W^{cs}(\epsilon)$, let $x_k = T^{kt_1}(x_0)$ for $k \ge 1$. Write x_k as

$$x_k = m_k + x_k^u + x_k^s$$

where $m_k \in M$, $x_k^\alpha \in X_{m_k}^\alpha(\epsilon)$ for $\alpha = u, s$. Then,

Proposition 4.8. *For $y_0^\alpha \in X_{m_0}^\alpha(3\epsilon)$, $\alpha = u, s, c$, $f_{x_0}^{ss}(y_0^s) = y_0^u + y_0^c$ if and only if*

$$T^{kt_1}(x_0 + y_0^u + y_0^s + y_0^c) = x_k + y_k^u + y_k^s + y_k^c,\ k \ge 1,$$

$|y_k^\alpha| \in X_{m_k}^\alpha(3\epsilon)$, for $\alpha = u, s, c$, and

$$|y_k^u| + |y_k^c| \le \mu |y_k^s|.$$

Proof. Suppose that $f_{x_0}^{ss}(y_0^s) = y_0^u + y_0^c$. The invariance of F^{ss} implies that

$$y_k^u + y_k^c = f_{x_k}^{ss}(y_k^s).$$

Hence, $|y_k^u| + |y_k^c| \le \mu |y_k^s|$ and $|y_k^\alpha| < 3\epsilon$.

Now we prove the converse. For each $k > 0$, since $|y_k^u| + |y_k^c| \le \mu |y_k^s|$, we may choose a stable fiber f_{x_k} such that

$$y_k^u + y_k^c = f_{x_k}(y_k^s).$$

Repeatedly using Lemma 4.3, we obtain a sequence of stable fibers, all depending upon k,

$$f_{x_{k-1}}, \dots, f_{x_0}$$

such that for $1 \le j \le k$,

$$T^{t_1}\big(\text{graph}(f_{x_{j-1}})\big) \subset T^{t_1}\big(\text{graph}(f_{x_j})\big),$$

and $f_{x_j}(y_j^s) = y_j^u + y_j^c$. By using Lemma 4.6, we have

$$\begin{aligned} &|y_0^u + y_0^c - f_{x_0}^{ss}(y_0^s)| \\ &= |f_{x_0}(y_0^s) - f_{x_0}^{ss}(y_0^s)| \le |y_0^s| \|f_{x_0} - f_{x_0}^{ss}\| \\ &\le |y_0^s|\lambda_1 \|f_{x_1} - f_{x_1}^{ss}\| \le |y_0^s|\lambda_1^k \|f_{x_k} - f_{x_k}^{ss}\| \\ &\le 2\mu |y_0^s| \lambda_1^k \to 0 \text{ as } k \to \infty, \end{aligned}$$

which yields

$$y_0^u + y_0^c = f_{x_0}^{ss}(y_0^s).$$

This completes the proof. □

For $x_0, \tilde{x}_0 \in W^{cs}(\epsilon)$, write

$$x_0 = m_0 + x_0^u + x_0^s = \tilde{x}_0 + \tilde{y}_0^u + \tilde{y}_0^s + \tilde{y}_0^c,$$

$$\tilde{x}_0 = \tilde{m}_0 + \tilde{x}_0^u + \tilde{x}_0^s = x_0 + y_0^u + y_0^s + y_0^c,$$

where

$$x_0^\alpha \in X_{m_0}^\alpha(\epsilon), \tilde{x}_0^\alpha \in X_{\tilde{m}_0}^\alpha(\epsilon) \text{ for } \alpha = u, s,$$

$$y_0^\alpha \in X_{m_0}^\alpha \text{ and } \tilde{y}_0^\alpha \in X_{\tilde{m}_0}^\alpha \text{ for } \alpha = u, s, c.$$

Then,

Proposition 4.9. *If $y_0^\alpha \in X_{m_0}^\alpha(3\epsilon)$ and $\tilde{y}_0^\alpha \in X_{\tilde{m}_0}^\alpha(3\epsilon)$ for $\alpha = u, s, c$, then $y_0^u + y_0^c = f_{x_0}^{ss}(y_0^s)$ if and only if $\tilde{y}_0^u + \tilde{y}_0^c = f_{\tilde{x}_0}^{ss}(\tilde{y}_0^s)$.*

Proof. It is enough to show that if $y_0^u + y_0^c = f_{x_0}^{ss}(y_0^s)$, then $\tilde{y}_0^u + \tilde{y}_0^c = f_{\tilde{x}_0}^{ss}(\tilde{y}_0^s)$.

For $k \ge 1$ let

$$\begin{aligned} x_k &= T^{kt_1}(x_0) = m_k + x_k^u + x_k^s, \\ \tilde{x}_k &= T^{kt_1}(\tilde{x}_0) = \tilde{m}_k + \tilde{x}_k^u + \tilde{x}_k^s. \end{aligned}$$

Note that $x_k^\alpha \in X_{m_k}^\alpha(\epsilon)$ and $\tilde{x}_k^\alpha \in X_{\tilde{m}_k}^\alpha(\epsilon)$ for $\alpha = u, s$. We write $\tilde{x}_k$ as

$$\tilde{x}_k = x_k + y_k^u + y_k^s + y_k^c. \tag{4.28}$$

It follows from the invariance of F^{ss} that

$$y_k^u + y_k^c = f_{x_k}^{ss}(y_k^s). \tag{4.29}$$

Let

$$x_k = \tilde{x}_k + \tilde{y}_k^u + \tilde{y}_k^s + \tilde{y}_k^c. \tag{4.30}$$

Add (4.28) to (4.30) to obtain

$$\tilde{y}_k^u + y_k^u + \tilde{y}_k^s + y_k^s + \tilde{y}_k^c + y_k^c = 0.$$

Applying the projection $\Pi_{\tilde{m}_k}^\alpha$ to the above identity, we have for $\alpha = u, s, c$

$$\tilde{y}_k^\alpha = -\Pi_{\tilde{m}_k}^\alpha (y_k^u + y_k^s + y_k^c). \tag{4.31}$$

Note that $|m_k - \tilde{m}_k| \leq 9\epsilon$. Let $\alpha = s$ in (4.31). Then, using (4.27), (4.29), we obtain

$$\begin{aligned}|\tilde{y}_k^s| &\leq |y_k^s| + O(\epsilon)(|y_k^u| + |y_k^s| + |y_k^c|)\\ &\leq \big(1 + O(\epsilon)\big)|y_k^s|\\ &\leq \lambda_1^k\big(1 + O(\epsilon)\big)|y_1^s|\\ &< 3\epsilon\end{aligned}$$

provided that ϵ is sufficiently small.

Let $\alpha = u, c$ in (4.31). We have

$$\begin{aligned}|\tilde{y}_k^\alpha| &\leq |y_k^\alpha| + O(\epsilon)(|y_k^u| + |y_k^s| + |y_k^c|)\\ &\leq \big(\mu + O(\epsilon)\big)|y_k^s|\\ &< \epsilon\end{aligned}$$

provided that ϵ is sufficiently small since $\mu < \frac{1}{3}$. Hence, $|\tilde{y}_k^\alpha| < 3\epsilon$ for $\alpha = u, s, c$.

To prove $\tilde{y}_0^u + \tilde{y}_0^c = f_{\tilde{x}_0}^{ss}(\tilde{y}_0^s)$, by Proposition 4.8, it is sufficient to prove

$$|\tilde{y}_k^u| + |\tilde{y}_k^c| \leq \mu|\tilde{y}_k^s|.$$

From (4.31), we obtain for $k \geq 1$,

$$\begin{aligned}|\tilde{y}_k^u| + |\tilde{y}_k^c| &\leq |y_k^u| + |y_k^c| + O(\epsilon)|y_k^s|\\ &\leq \big(\lambda_1\mu + O(\epsilon)\big)|y_k^s|,\end{aligned} \tag{4.32}$$

where Lemma 4.4 is used. Again, from (4.31) we obtain

$$|\tilde{y}_k^s| \geq \big(1 - O(\epsilon)\big)|y_k^s|.$$

Therefore, (4.32) yields

$$|\tilde{y}^u| + |\tilde{y}^c| \leq \frac{\lambda_1\mu + O(\epsilon)}{1 - O(\epsilon)}|\tilde{y}_k^s| \leq \mu|\tilde{y}_k^s|$$

provided that ϵ is sufficiently small. This completes the proof of the proposition. □

Proposition 4.10. *For $x_0 = m_0 + x_0^u + x_0^s \in W^{cs}(\epsilon)$, there exists a unique $y_0^s \in X_{m_0}^s(3\epsilon)$ such that*

$$x_0 + y_0^s + f_{x_0}^{ss}(y_0^s) = m \in M$$

and $x_0 \in$ graph(f_m^{ss}).

Proof. We first prove the uniqueness. Suppose there exist $y_1^s, y_2^s \in X_{m_0}^s(3\epsilon)$ such that

$$x_0 + y_i^s + f_{x_0}^{ss}(y_i^s) = m_i \in M, \qquad i = 1, 2.$$

Clearly, $|m_0 - m_i| \leq 9\epsilon$. Write $f_{x_0}^{ss}(y_i^s) = y_i^u + y_i^c$. By Lemma 3.2, we have

$$|y_1^c - y_2^c - (m_1 - m_2)| \leq O(\epsilon_1)|m_1 - m_2|$$

and since $f_{x_0}^{ss}$ is Lipschitz

$$\begin{aligned}|y_1^s - y_2^s| &\leq O(\epsilon_1)|m_1 - m_2|\\ &\leq O(\epsilon_1)|y_1^s - y_2^s|.\end{aligned}$$

By choosing ϵ_1 small enough, we have $y_1^s = y_2^s$, which implies $m_1 = m_2$.

Suppose that there exists $y_0^s \in X_{m_0}^s(3\epsilon)$ such that $x_0 + y_0^s + f_{x_0}^{ss}(y_0^s) = m \in M$. Write

$$x_0 = m + y^s + y^u + y^c.$$

Applying Lemma 3.2 to m and x_0, we obtain

$$|y^c - (m_0 - m)| \le O(\epsilon_1)|m_0 - m|,$$

which gives

$$|m_0 - m| \le \big(1 + O(\epsilon_1)\big)|y^c|.$$

For $\alpha = u, s$, we have

$$\begin{aligned}|y^\alpha| &\le O(\epsilon_1)|y^c| + \|(\Pi_{m_0}^\alpha|_{X_m^u})^{-1}\||x_0^\alpha| \\ &\le O(\epsilon_1)\epsilon + \big(1 + O(\epsilon)\big)\epsilon < 2\epsilon,\end{aligned}$$

provided ϵ_1 and ϵ are sufficiently small.

Note that $y^s + y^u + y^c = -\big(y_0^s + f_{x_0}^{ss}(y_0^s)\big) = -(y_0^s + y_0^u + y_0^c)$. Thus,

$$\begin{aligned}|y^c| &= |\Pi_m^c(y_0^s + y_0^u + y_0^c)| \\ &\le O(\epsilon)(|y_0|^s + |y_0^u| + |y_0^c|) + |y_0^c| \\ &\le O(\epsilon)\epsilon + \epsilon \le 3\epsilon,\end{aligned}$$

provided that ϵ is sufficiently small.

It follows from Proposition 4.9 that

$$y^u + y^c = f_m^{ss}(y^s).$$

Finally, we show the existence of y_0^s.

For $y^s \in X_{m_0}^s(3\epsilon)$. We write $f_{x_0}^{ss}(y^s) = y^u + y^c$ and

$$x_0 + y^s + f_{x_0}^{ss}(y^s) = m + \tilde{y}^u + \tilde{y}^s,$$

where $\tilde{y}^\alpha \in X_m^\alpha(\epsilon_1)$ for $\alpha = u, s$, by Lemma 3.6. Note that $|m - m_0| \le r$. Here ϵ_1 and r are given in Section 3. Thus, from the choice of r, $\bar{y}^\alpha = (\Pi_m^\alpha|_{X_{m_0}^\alpha})^{-1}\tilde{y}^\alpha$ is well defined.

Define a map $\xi : \overline{X_{m_0}^s(3\epsilon)} \to X_{m_0}^s$ by

$$\xi(y^s) = y^s - \bar{y}^s.$$

Applying Lemma 3.2 to $m + \tilde{y}^s + \tilde{y}^u$ and m_0, we obtain

$$|y^c - (m - m_0)| \le O(\epsilon_1)|m - m_0|,$$

which implies

$$\begin{aligned}|m - m_0| &\le \big(1 + O(\epsilon_1)\big)|y^c| \\ &\le \mu\big(1 + O(\epsilon_1)\big)|y^s| \\ &\le \epsilon,\end{aligned}$$

provided ϵ_1 is sufficiently small. For $\alpha = u, s$, we have

$$|\bar{y}^\alpha - (y^\alpha + x_0^\alpha)| \le O(\epsilon_1)|y^c|,$$

and hence,

$$|\bar{y}^\alpha| \le 5\epsilon,$$

provided that ϵ_1 is small enough. In particular,

$$|\bar{y}^s - y^s| \le |x_0^\alpha| + O(\epsilon_1)|y^c| < 3\epsilon.$$

So ξ maps $\overline{X_{m_0}^s(3\epsilon)}$ into $X_{m_0}^s(3\epsilon)$.

Next, we show that ξ is a contraction. For $y_i^s \in \overline{X_{m_0}^s(3\epsilon)}$, $i = 1, 2$, let

$$x_0 + y_i^s + f_{x_0}^{ss}(y_i^s) = m_i + \tilde{y}_i^u + \tilde{y}_i^s$$

and

$$\bar{y}_i^\alpha = (\Pi_{m_i}|_{X_{m_0}^\alpha})^{-1}\tilde{y}_i^\alpha \text{ for } \alpha = u, s.$$

By Lemma 3.2, we have

$$\begin{aligned} &|y_1^s - y_2^s - (\bar{y}_1^s - \bar{y}_2^s)| + |y_1^u - y_2^u - (\bar{y}_1^u - \bar{y}_2^u)| \\ &\le O(\epsilon_1)|y_1^c - y_2^c| + O(\epsilon)(|\bar{y}_1^s - \bar{y}_2^s| + |\bar{y}_1^u - \bar{y}_2^u|), \end{aligned} \tag{4.33}$$

which gives

$$\begin{aligned} &|\bar{y}_1^s - \bar{y}_2^s| + |\bar{y}_1^u - \bar{y}_2^u| \\ &\le O(\epsilon_1)|y_1^c - y_2^c| + \big(1 + O(\epsilon)\big)\big(|y_1^s - y_2^s| + |y_1^u - y_2^u|\big). \end{aligned}$$

Hence, from (4.33) and the Lipschitz condition, it follows that

$$\begin{aligned} |\xi(y_1^s) - \xi(y_2^s)| =& |y_1^s - y_2^s - (\bar{y}_1^s - \bar{y}_2^s)| \\ \le& \big(O(\epsilon_1) + O(\epsilon)\big)|y_1^s - y_2^s| \\ \le& \frac{1}{2}|y_1^s - y_2^s|, \end{aligned}$$

provided that ϵ_1 and ϵ are sufficiently small.

Therefore, ξ is a contraction map and has a unique fixed point y_0^s, which yields, $\bar{y}_0^s = 0$ and hence, $\tilde{y}_0^s = 0$. The fact that $m + \tilde{y}_0^u \in W^{cs}(\epsilon)$ locally a Lipschitz graph over $X_m^c \oplus X_m^s$, implies that $\tilde{y}_0^u = 0$. Thus, $m = x_0 + y_0^s + f_{x_0}^{ss}(y_0^s)$. The proof is complete. □

Proposition 4.11. *For each $x_0 \in W^{cs}(\epsilon)$, there exists a unique $m \in M$ such that $x_0 \in$ graph(f_m^{ss}).*

Proof. Proposition 4.10 gives the existence of m. Suppose that there exist $m_1, m_2 \in M$ such that

$$x_0 \in \text{ graph}(f_{m_i}^{ss}), \qquad i = 1, 2.$$

Then $x_0 = m_i + y_i^s + f_{m_i}^{ss}(y_i^s)$, $i = 1, 2$, where $y_i^s \in X_{m_i}(3\epsilon)$.

Let $y_i^u + y_i^c = f_{m_i}^{ss}(y_i^s)$. Note that $|m_0 - m_i| \le 6\epsilon$. Applying Lemma 3.2 to x_0 and m_i, we obtain

$$|y_i^s| \le O(\epsilon_1)\epsilon + \big(1 + O(\epsilon)\big)|x_0^\alpha| \le 2\epsilon,$$

provided ϵ_1 and ϵ are sufficiently small. Therefore, $|y_i^\alpha| < 3\epsilon$ for $\alpha = u, s$, and c.

Let

$$m_i = x_0 + \bar{y}_i^u + \bar{y}_i^s + \bar{y}_i^c,$$

where $\bar{y}_i^\alpha \in X_{m_0}^\alpha$. Then,

$$\bar{y}_i^u + \bar{y}_i^s + \bar{y}_i^c + y_i^u + y_i^s + y_i^c = 0.$$

Thus, for $\alpha = u, s, c$,

$$|\bar{y}_i^\alpha| = O(\epsilon)(|y_i^u| + |y_i^s| + |y_i^c|) + |y_i^\alpha| < 3\epsilon$$

provided ϵ is small enough.

By Proposition 4.9, $m_1, m_2 \in$ graph$(f_{x_0}^{ss})$, hence $m_1 = m_2$ from Proposition 4.10. The proof is complete. □

From this proposition, we may define a projection from $W^{cs}(\epsilon)$ to M by

$$P^s(x_0) = m.$$

Lemma 4.12. *P^s is continuous.*

Proof. Suppose this is not true. Then there exist a constant c and a sequence

$$x_n = m_n + x_n^s + f_{m_n}^{ss}(x_n^s),$$

where $x_n^s \in X_{m_n}^s(3\epsilon)$, such that $x_n \to x_0$, but

$$|m_n - m_0| > c > 0,$$

where $m_0 = P^s(x_0)$. Since M is compact, we may assume $m_n \to \tilde{m}_0 \in M$. Write

$$x_n = \tilde{m}_0 + \tilde{x}_n^u + \tilde{x}_n^s + \tilde{x}_n^c.$$

Then $\tilde{x}_n^\alpha \to \tilde{x}_0^\alpha \in X_{\tilde{m}_0}^\alpha$ as $n \to \infty$. From the continuity of f_x^{ss}, we have $\tilde{x}_0^u + \tilde{x}_0^c = f_{\tilde{m}_0}^{ss}(\tilde{x}_0^s)$, a contradiction to Proposition 4.10. The proof is complete. ☐

Proposition 4.13. *For all $m \in M$ and $t > 0$, $T^t(graph(f_m^{ss})) \cap \Theta\big(X^u(\epsilon) \oplus X^s(\epsilon)\big) \subset graph(f_{T^t m}^{ss})$.*

Proof. For possibly much smaller ϵ and $x \in \text{graph}(f_m^{ss})$, we write

$$x_t := T^t(x) = \bar{m}_t + \bar{x}_t^s + \bar{x}_t^u + \bar{x}_t^c$$

where $\bar{x}_t^u + \bar{x}_t^c = f_{\bar{m}_t}^{ss}(\bar{x}_t^s)$. The reason we may need smaller ϵ here is to guarantee that x_t can be written in the above form.

Let $m_t = T^t(m)$. We also write

$$x_t = m_t + x_t^s + x_t^u + x_t^c,$$

where $x_t^\alpha \in X_{m_t}^\alpha$ for $\alpha = u, s$, and c.

Note that, because of the invariance under T^{t_1}, the two representations coincide for integer multiples of t_1; that is,

$$\bar{x}_{kt_1}^\alpha = x_{kt_1}^\alpha$$

for all integers $k > 0$ and $\alpha = u, s$, and c. Thus,

$$|x_{kt_1}^u| + |x_{kt_1}^c| \leq \mu |x_{kt_1}^s|. \tag{4.34}$$

It is easy to see that it is enough to show that

$$m_t = \bar{m}_t \text{ for } t \in (0, t_1). \tag{4.35}$$

We prove this by a contradiction. Suppose (4.35) does not hold. Then, there exists $t_2 \in (0, t_1)$ such that

$$m_{t_2} \neq \bar{m}_{t_2}.$$

Since T^t is a diffeomorphism from M onto M, $m_{kt_1+t_2} \neq \bar{m}_{kt_1+t_2}$ for all integers $k > 0$.

By (4.27), we have

$$|\bar{x}_{kt_1}^s| \leq 3\lambda_1^k \epsilon.$$

For k so large that x_{kt_1} is sufficiently close to m_{kt_1}, we obtain

$$\begin{aligned} &|x_{kt_1+t_2} - m_{kt_1+t_2}| \\ &= |T^{t_2}(x_{kt_1}) - T^{t_2}(m_{kt_1})| \\ &\leq \bar{C}|x_{kt_1} - m_{kt_1}| \leq \bar{C}\lambda_1^k \epsilon, \end{aligned}$$

where $\bar{C}$ depends on t_2. Here the fact that $m_{kt_1} = \bar{m}_{kt_1}$ is used.

Hence,

$$|x^{\alpha}_{kt_1+t_2}| \le \bar{C}\lambda_1^k \epsilon.$$

Choose k_0 so large that $C\lambda_1^{k_0}\epsilon < \epsilon$. From Proposition 4.8, there exists $k_1 > k_0$ such that

$$|x^u_{kt_1+t_2}| + |x^c_{kt_1+t_2}| > \mu |x^s_{kt_1+t_2}| \tag{4.36}$$

for all integers $k \ge k_1$.

On the other hand, since $T^{2t_1-t_2}$ satisfies (2.1) and (2.2), using Lemma 3.3 and (4.36), we obtain

$$|x^u_{(k_1+2)t_1}| + |x^c_{(k_1+2)t_1}| > \mu |x^s_{(k_1+2)t_1}|,$$

which contradicts (4.34). This completes the proof. □

Proposition 4.14. (i) *For $x, y \in W^{ss}_m(\epsilon)$, we have*

$$|T^t(x) - T^t(y)| \to 0$$

exponentially as $t \to +\infty$. (ii) *For $x \in W^{ss}_m(\epsilon)$, $\bar{m} \in M$, and $\bar{m} \ne m$, we have*

$$\frac{|T^t(x) - T^t(m)|}{|T^t(x) - T^t(\bar{m})|} \to 0$$

exponentially as $t \to +\infty$.

Proof. For $x = m + x^u + x^s + x^c \in W^{ss}_m(\epsilon)$, from the invariance of the stable foliation for T^{t_1}, we can write

$$T^{kt_1}(x) = m_k + x^u_k + x^s_k + x^c_k$$

where $m_k = T^{kt_1}(m)$, $x^\alpha_k \in X^\alpha_{m_k}$ for $\alpha = u, s, c$, and $x^u_k + x^c_k = f^{ss}_{m_k}(x^s_k)$ with $|x^s_k| < 3\epsilon$. It follows from (4.27) that

$$|x^s_k| \le \lambda_1^k |x^s| \quad \text{for all } k \ge 0, \tag{4.37}$$

which together with $|x^u_k| + |x^c_k| \le \mu |x^s_k|$ gives

$$|T^{kt_1}(x) - T^{kt_1}(m)| = |x^u_k + x^c_k + x^s_k| \le (1+\mu)\lambda_1^k |x^s|. \tag{4.38}$$

For each $t > t_1$, we write $t = kt_1 + t_2$ for some $k > 0$ and $t_2 \in [0, t_1)$. By using (4.38) and Proposition 4.13, we have

$$\begin{aligned} &|T^t(x) - T^t(m)| \\ &= |T^{kt_1}\big(T^{t_2}(x)\big) - T^{kt_1}\big(T^{t_2}(m)\big)| \\ &\le 3\epsilon(1+\mu)\lambda_1^k \le Ce^{-\omega t}, \end{aligned}$$

where $\omega = -(\ln\ \lambda_1)/t_1 > 0$, and $C = 3\epsilon(1+\mu)e^{\omega t_2}$. This completes the proof of (i).

Since M is compact and T^{t_1} is a diffeomorphism from M onto M, for a given ϵ_1 there are $0 < r_1 < r_2$ such that $B(m, r_i), i = 1, 2$, are $\eta-$neighborhoods satisfying

$$M \cap B(T^{t_1}(m), r_1) \subset T^{t_1}\big(M \cap B(m, r_2)\big), \tag{4.39}$$

$$T^{t_1}\big(M \cap B(m, r_1)\big) \subset M \cap B(T^{t_1}(m), r_2), \tag{4.40}$$

and such that Lemma 3.2 can be applied to these $\eta-$neighborhoods.

For $\bar{m} \in M$, denote $\bar{m}_k = T^{kt_1}(\bar{m})$ for $k = 0, 1, \cdots$. Suppose that $\bar{m}_{k+1} \in B(m_{k+1}, r_1)$ which is an $\eta-$neighborhood of m_k. For $i = k-1, k$, we write

$$\bar{m}_i = m_i + \bar{x}_i^u + \bar{x}_i^s + \bar{x}_i^c.$$

Because of (4.39), we can apply Lemma 3.2 to obtain, for $i = k-1, k$,

$$|\bar{m}_i - m_i| \leq C|\bar{x}_i^c| \tag{4.41}$$

and

$$|\bar{x}_i^u| + |\bar{x}_i^s| \leq C\epsilon_1 |\bar{x}_i^c|. \tag{4.42}$$

Thus, by choosing ϵ_1 and ϵ^* sufficiently small, we obtain

$$\frac{|x_k^s|}{|\bar{x}_k^c|} \leq \lambda_1 \frac{|x_{k-1}^s|}{|\bar{x}_{k-1}^c|}. \tag{4.43}$$

By using (4.41), (4.42) and $|x_i^u + x_i^c| \leq \mu |x_i^s|$ and choosing ϵ_1 small enough, we obtain for $i = k-1, k$

$$\frac{1}{2}\frac{|x_i^s|}{|\bar{x}_i^c|} \leq \frac{|x_i^u + x_i^s + x_i^c|}{|\bar{x}_i^u + \bar{x}_i^s + \bar{x}_i^c|} \leq 2\frac{|x_i^s|}{|\bar{x}_i^c|}. \tag{4.44}$$

For each $k > 0$, we consider two cases:

Case 1: $\bar{m}_i \in B(m_i, r_1)$ for all $i = 0, 1, \cdots, k$.

Case 2: There is a $0 \leq k_0 \leq k+1$ such that $\bar{m}_i \notin B(m_i, r_1)$ for $i = k_0+1, \cdots, k$, but $\bar{m}_{k_0} \in B(m_{k_0}, r_1)$.

For Case 1, it follows from (4.43) and (4.44) that

$$\begin{aligned} &\frac{|T^{kt_1}(x) - T^{kt_1}(m)|}{|T^{kt_1}(\bar{m}) - T^{kt_1}(m)|} \\ &\leq 2\frac{|x_k^s|}{|\bar{x}_k^c|} \leq 2\lambda_1^k \frac{|x^s|}{|\bar{x}^c|} \leq \frac{12(1+\mu)\epsilon}{\bar{m}-m}\lambda_1^k. \end{aligned} \tag{4.45}$$

For Case 2, again from (4.43) and (4.44), by using (4.37), we obtain

$$\begin{aligned} &\frac{|T^{kt_1}(x) - T^{kt_1}(m)|}{|T^{kt_1}(\bar{m}) - T^{kt_1}(m)|} \\ &\leq 2\frac{|x_k^s|}{|\bar{x}_k^c|} \leq 2\lambda_1^{k-k_0}\frac{|x_{k_0}^s|}{|\bar{x}_{k_0}^c|} \\ &\leq 4\lambda_1^{k-k_0}\frac{|T^{(k_0)t_1}(x) - T^{(k_0)t_1}(m)|}{|T^{(k_0)t_1}(\bar{m}) - T^{(k_0)t_1}(m)|} \\ &\leq \frac{12(1+\mu)\epsilon}{r_1}\lambda_1^k. \end{aligned} \tag{4.46}$$

Combining (4.45) and (4.46) gives for all $k > 0$

$$\frac{|T^{kt_1}(x) - T^{kt_1}(m)|}{|T^{kt_1}(\bar{m}) - T^{kt_1}(m)|} \leq \frac{12(1+\mu)\epsilon}{\min\{r_1, \bar{m}-m\}}\lambda_1^k,$$

which together with the fact that $\inf\{|T^t(\bar{m}) - T^t(m)| : 0 \leq t \leq t_1\} > 0$ yields

$$\frac{|T^t(x) - T^t(m)|}{|T^t(\bar{m}) - T^t(m)|} \to 0, \qquad \text{exponentially as } t \to \infty.$$

Hence, using statement (i) in this proposition, we obtain (ii). This completes the proof. □

Combining all propositions together, we obtain Theorem 4.1.

5. C^1 SMOOTHNESS OF THE STABLE FOLIATION

In this section, we prove the smoothness of $f_m^{ss}(y^s)$ in y^s. We take the approach used in [BLZ1]. The basic idea is to find a candidate for the tangent bundle of the stable fiber, which is invariant under the linearization DT^{t_1}, then to prove it is tangent to the stable fiber. The arguments are based on the idea of Lipschitz jets, as in [HPS], but with a Lipschitz jet space which is different from that introduced in [HPS].

Let Y and Z be Banach spaces. For $y_0 \in Y$ and $z_0 \in Z$, consider two local maps

$$g_i : U_i \to Z, \quad g_i(y_0) = z_0, \qquad i = 1, 2,$$

where U_i is a neighborhood of y_0.

Define

$$d(g_1, g_2) = \overline{\lim_{y \to y_0}} \frac{|g_1(y) - g_2(y)|}{|y - y_0|}.$$

We shall see that $d(g_1, g_2)$ defines a metric on a certain quotient space.

If $d(g_1, g_2) = 0$, we say that g_1 is equivalent to g_2. The equivalence class of all local maps equivalent to g_1 is called the Lipschitz jet of g_1 at y_0, which is denoted by $j_1 = [g_1]$. We use $J(Y, Z; y_0, z_0)$ to denote the set of all jets at y_0 carrying y_0 to z_0. For $j_1, j_2 \in J(Y, Z; y_0, z_0)$, we define

$$d(j_1, j_2) = \overline{\lim_{y \to y_0}} \frac{|g_1(y) - g_2(y)|}{|y - y_0|}$$

where g_1 and g_2 are representatives of j_1 and j_2, respectively. It is not hard to see that $d(j_1, j_2)$ does not depend on the choices of the representatives.

Consider the jet spaces

$$\begin{aligned} J^b &= \Big\{ j \in J(Y, Z; y_0, z_0) : \ d(j, [z_0]) < \infty \Big\}, \\ J^c &= \Big\{ j \in J^b : \ j \text{ has a representative which is continuous} \\ &\qquad\qquad\qquad\qquad \text{in a neighborhood of } y_0 \Big\}, \\ J^d &= \Big\{ j \in J^b : \ j \text{ has a differentiable representative} \Big\}, \\ J^a &= \Big\{ j \in J^b : \ j \text{ has an affine representative} \Big\}. \end{aligned}$$

Theorem on Lipschitz Jets. *Let $y_0 = 0$, $z_0 = 0$. Then J^b is a Banach space with norm $\|j\| = d(j, 0)$. The sets J^c, J^d and J^a are closed subspaces of J^b and*

$$J^b \supset J^c \supset J^d = J^a.$$

The above results are borrowed from [HPS].

For each $m \in M$, set

$$\begin{aligned} J^b(m) &= J^b(X_m^s, X_m^c; 0, 0), \\ J^c(m) &= J^c(X_m^s, X_m^c; 0, 0), \\ J^d(m) &= J^d(X_m^s, X_m^c; 0, 0), \\ J^a(m) &= J^a(X_m^s, X_m^c; 0, 0). \end{aligned}$$

Let $\rho \in (1,2)$ be such that $\theta = \mu\rho < \frac{1}{2}$. For each fixed $m \in M$, we define a Lipschitz jet space

$$J_\theta^\ell(m) = J_\theta^\ell(X_m^s, X_m^c; 0, 0)$$

to be the space of jets which have a representative with Lipschitz constant θ in a neighborhood of 0. Let $J_\theta^\ell = J_\theta^\ell(X^s, X^c; 0, 0)$ denote the Lipschitz jet bundle over M with fiber $J_\theta^\ell(m)$.

Consider all maps from $W^{cs}(\epsilon)$ to J_θ^ℓ which map points $x = m + x^s + f_m^{ss}(x^s)$ to jets j in $J_\theta^\ell(m)$.

Define

$$\Sigma_\theta^{ss,\ell} = \Big\{\gamma : W^{cs}(\epsilon) \to J_\theta^\ell \text{ with } \|\gamma\| < \infty\Big\}$$

where $\|\gamma\| = \sup\Big\{\|\gamma(x)\| : x \in W^{cs}(\epsilon)\Big\}$. Similarly, we may define $\Sigma_\theta^{ss,d}$ and $\Sigma_\theta^{ss,a}$.

For $x = m + x^s + f_m^{ss}(x^s) \in W^{cs}(\epsilon)$, from the invariance of F^{ss}, we may write

$$T^{t_1}(x) = \tilde{x} = \tilde{m} + \tilde{x}^s + f_{\tilde{m}}^{ss}(\tilde{x}^s), \tag{5.1}$$

where $\tilde{x}^s \in X_{\tilde{m}}^s(3\epsilon)$ and $\tilde{m} = T^{t_1}m$.

We also write

$$f_m^{ss}(x^s) = x^u + x^c$$

and

$$f_{\tilde{m}}^{ss}(\tilde{x}^s) = \tilde{x}^u + \tilde{x}^c.$$

Note that $x^\alpha \in X_m^\alpha(\epsilon)$ and $\tilde{x}^\alpha \in X_{\tilde{m}}^\alpha(\epsilon)$ for $\alpha = u, c$.

Let $\tilde{\gamma} \in \Sigma_\theta^{ss,\ell}$ be fixed and let $\tilde{g} : X_{\tilde{m}}^s(\tilde{r}) \to X_{\tilde{m}}^c$ be a Lipschitz representative of $\tilde{\gamma}(\tilde{x})$ such that for $y_1^s, y_2^s \in \tilde{X}_{\tilde{m}}^s(\tilde{r})$,

$$|\tilde{g}(y_1^s) - \tilde{g}(y_2^s)| \le \theta|y_1^s - y_2^s|, \tag{5.2}$$

where $\tilde{r}$ is a positive constant.

We shall construct $\gamma \in \Sigma_\theta^{ss,\ell}$ such that $\tilde{\gamma}$ is the image of γ under DT^{t_1} in a certain sense, which will be stated precisely, later. For a given $\tilde{g}$ satisfying the above properties, we want to find a Lipschitz map $g : X_m^s(r) \to X_m^c$ for some $r > 0$ such that for each $y^s \in X_m^s(r)$, there exists $\tilde{y}^s \in X_{\tilde{m}}^s(\tilde{r})$ satisfying

$$DT^{t_1}(x)\big(y^u + y^s + g(y^s)\big) = \tilde{y}^u + \tilde{y}^s + \tilde{g}(\tilde{y}^s), \tag{5.3}$$

where $y^u = Df_m^{cs}(x^s, x^c)(y^s, y^c)$, $y^c = g(y^s)$, $\tilde{y}^u = Df_{\tilde{m}}^{cs}(x^s, x^c)(\tilde{y}^s, \tilde{y}^c)$, $\tilde{y}^c = \tilde{g}(\tilde{y}^s)$, and f_m^{cs} is the C^1 local representative of $W^{cs}(\epsilon)$ defined on $X_m^c(\rho^{-1}\epsilon_1) \oplus X_m^s(\rho^{-1}\epsilon_1)$ with Lipschitz constant θ, and $\tilde{f}_{\tilde{m}}^{cs}$ is similar. To see this, for $y^s \in X_m^s(r)$, we define a map $E : \overline{X_m^c(r)} \to X_m^c$ for some small $r > 0$, by

$$\begin{aligned} E(y^c) = \Big(\Pi_{\tilde{m}}^c DT^{t_1}(x)|_{X_m^c}\Big)^{-1}\Big(&\tilde{g}\big(\Pi_{\tilde{m}}^s DT^{t_1}(x)(y^u + y^s + y^c)\big)\Big) \\ &- \Pi_{\tilde{m}}^c DT^{t_1}(x)(y^u + y^s)\Big) \end{aligned} \tag{5.4}$$

where $y^u = Df_m^{cs}(x^s, x^c)(y^s, y^c)$.

We shall see that E is a contraction and its fixed point gives a solution of (5.3).

Lemma 5.2. *There exists $\epsilon^* > 0$ such that for $\epsilon < \epsilon^*$ if $x = m + x^s + f_m^{ss}(x^s) \in W^{cs}(\epsilon)$ and $\tilde{g}$ satisfies (5.2), then there exists $r^* = r^*(\tilde{r}) > 0$ such that for $r < r^*$, E is a contraction on $\overline{X_m^c(r)}$.*

Proof. We first show that E is well defined from $\overline{X^c_m(r)}$ into itself. Observe that for $y^\alpha \in \overline{X^\alpha_m(r)}$, $\alpha = u, s, c$,

$$|\Pi^s_{\tilde{m}} DT^{t_1}(x)(y^u + y^s + y^c)| \le Cr \le \tilde{r}$$

provided that $r^* < \tilde{r}/C$. Thus, $\tilde{g}\big(\Pi^s_{\tilde{m}} DT^{t_1}(x)(y^u + y^s + y^c)\big)$ is well defined. From Lemma 3.5, $(\Pi^c_{\tilde{m}} DT^{t_1}(x)|_{X^c_m})^{-1}$ exists. Hence, $E(y^c)$ is well defined for $y^c \in \overline{X^c_m(r)}$.

Next, we show $|E(y^c)| < r$. Using (5.2) and Lemma 3.5, we obtain

$$\begin{aligned}|E(y^c)| &\le \|(\Pi^c_{\tilde{m}} DT^{t_1}(x)|_{X^c_m})^{-1}\|(\theta|\Pi^s_{\tilde{m}} DT^{t_1}(x)(y^u + y^s + y^c)| \\ &\qquad + |\Pi^c_{\tilde{m}} DT^{t_1}(x)(y^s + y^u)|) \\ &\le \|(\Pi^c_{\tilde{m}} DT^{t_1}(x)|_{X^c_m})^{-1}\|(O(\epsilon) + \theta\|\Pi^s_{\tilde{m}} DT^{t_1}(x)|_{X^s_m}\|)r \\ &\le (O(\epsilon) + \theta\lambda)r < r\end{aligned}$$

provided that ϵ^* is sufficiently small. Here $0 < \lambda < 1$ and $\theta < \frac{1}{2}$ are used.

Finally, we show that E is a contraction. For $y^c_1, y^c_2 \in \overline{X^c_m(r)}$, from (5.2), (5.4) and Lemma 3.5, we have

$$\begin{aligned}&|E(y^c_1) - E(y^c_2)| \\ &\le \|(\Pi^c_{\tilde{m}} DT^{t_1}(x)|_{X^c_m})^{-1}\|(\theta|\Pi^s_{\tilde{m}} DT^{t_1}(x)(y^u_1 - y^u_2 + y^c_1 - y^c_2)| \\ &\qquad + |\Pi^c_{\tilde{m}} DT^{t_1}(x)(y^u_1 - y^u_2)|) \\ &\le O(\epsilon)(|y^u_1 - y^u_2| + |y^c_1 - y^c_2|) \\ &\le O(\epsilon)|y^c_1 - y^c_2| \le \frac{1}{2}|y^c_1 - y^c_2|,\end{aligned}$$

provided that ϵ^* is sufficiently small. This completes the proof. □

By the contraction mapping theorem, we obtain that for each $y^s \in \underline{X^s_m(r)}$, E has a unique fixed point $y^c \in \overline{X^c_m(r)}$, which defines a map from $X^s_m(r)$ to $\overline{X^c_m(r)}$. We denote it by $y^c = g(y^s)$. Clearly, g satisfies (5.3). Furthermore, this function is a Lipschitz function with the Lipschitz constant θ.

Lemma 5.3. *There exists $\epsilon^* > 0$ such that for $\epsilon < \epsilon^*$ if $x \in W^{cs}(\epsilon)$, then for $y^s_1, y^s_2 \in X^s_m(r)$,*

$$|g(y^s_1) - g(y^s_2)| \le \theta|y^s_1 - y^s_2|. \tag{5.5}$$

Proof. From the definition of g, using Lemma 3.5, it follows that

$$\begin{aligned}&|g(y^s_1) - g(y^s_2)| \\ &\le \|(\Pi^c_{\tilde{m}} DT^{t_1}(x)|_{X^u_m})^{-1}\|(\theta|\Pi^s_{\tilde{m}} DT^{t_1}(x)(y^u_1 - y^u_2 + y^s_1 - y^s_2 + y^c_1 - y^c_2)| \\ &\qquad + |\Pi^c_{\tilde{m}} DT^{t_1}(x)(y^u_1 - y^u_2 + y^s_1 - y^s_2)|) \\ &\le O(\epsilon)|y^c_1 - y^c_2| + (O(\epsilon) + \theta\lambda)|y^s_1 - y^s_2|,\end{aligned}$$

which implies that

$$|g(y^s_1) - g(y^s_2)| = |y^c_1 - y^c_2| \le \frac{O(\epsilon) + \theta\lambda}{1 - O(\epsilon)}|y^s_1 - y^s_2| \le \theta|y^s_1 - y^s_2|$$

by choosing ϵ^* small enough. The proof is complete. □

Next, we want to show that the jet equivalence class $[g]$ does not depend on the choice of $\tilde{g} \in \tilde{\gamma}(\tilde{x})$. Let $\tilde{h}$ be another Lipschitz representative of $\tilde{\gamma}(\tilde{x})$ satisfying

$$|\tilde{h}(\tilde{y}_1^s) - \tilde{h}(\tilde{y}_2^s)| \le \theta |\tilde{y}_1^s - \tilde{y}_2^s|$$

for $\tilde{y}_1^s, \tilde{y}_2^s \in X_{\tilde{m}}^s(\tilde{r}_0)$, where $\tilde{r}_0$ is a positive constant. In the same way as above, define $h : X_m^s(r_0) \to X_m^c(r_0)$.

Lemma 5.4. $[g] = [h]$.

Proof. Let $y^s \in X_m^s(\hat{r})$, where $\hat{r} = \min\Big\{r, r_0\Big\}$. Set

$$y_1^c = g(y^s),\ y_2^c = h(y^s).$$

We want to show that

$$\|[g] - [h]\| = \overline{\lim}_{y^s \to 0} \frac{|g(y^s) - h(y^s)|}{|y^s|} = 0.$$

Note that from $[\tilde{g}] = [\tilde{h}]$, $|\tilde{g}(\tilde{y}^s) - \tilde{h}(\tilde{y}^s)| = o(|\tilde{y}^s|)$ as $\tilde{y}^s \to 0$. From the definition of g and h, by Lemma 3.5, we obtain

$$|y_1^c - y_2^c| \le O(\epsilon)(|y_1^c - y_2^c|) + o(|\Pi_{\tilde{m}}^s DT^{t_1}(x)(y_1^u + y^s + y_1^c)|),$$

which, together with $|y_1^c| \le \theta |y^s|$ and $|y_1^u| \le \theta(|y^s| + |y_1^c|)$, implies that

$$|y_1^c - y_2^c| = o(|y^s|).$$

The proof is complete. □

Thus, for each given $\tilde{\gamma} \in \Sigma_\theta^{ss,\ell}$ one may define $\gamma \in \Sigma_\theta^{ss,\ell}$ by $\gamma(x) = [g]$.

The following summarizes what we have so far:

Proposition 5.5. *There exists ϵ^* such that if $\epsilon < \epsilon^*$, then for each $\tilde{\gamma} \in \Sigma_\theta^{ss,\ell}$, there exists a unique $\gamma \in \Sigma_\theta^{ss,\ell}$ such that DT^{t_1} maps γ to $\tilde{\gamma}$ in the sense that (5.3) holds.*

Define

$$\mathcal{F}(\tilde{\gamma}) = \gamma.$$

We claim

Lemma 5.6. *$\mathcal{F}$ is a contraction on $\Sigma_\theta^{ss,\ell}$.*

Proof. Let $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \Sigma_\theta^{ss,\ell}$. For $x = m + x^s + f_m^{ss}(x^s)$, we again write

$$T^{t_1}(x) = \tilde{x} = \tilde{m} + \tilde{x}^s + f_{\tilde{m}}^{ss}(\tilde{x}^s).$$

By Proposition 5.5, there exist Lipschitz functions $g_1, g_2 : X_m^s(r) \to X_m^c(r)$ such that for $i = 1, 2$,

$$|g_i(y_1^s) - g_i(y_2^s)| \le \theta |y_1^s - y_2^s| \tag{5.6}$$

and

$$[g_1] = \mathcal{F}(\tilde{\gamma}_1)(x),$$
$$[g_2] = \mathcal{F}(\tilde{\gamma}_2)(x).$$

Let $y_i^c = g_i(y^s)$ and $y_i^u = Df_m^{cs}(x^s, x^c)(y^s, y_i^c)$ for $i = 1, 2$. From the definition of g_1 and g_2

$$\begin{aligned}
g_1(y^s) - g_2(y^s) &= y_1^c - y_2^c \\
&= \Big(\Pi_{\tilde m}^c DT^{t_1}(x)|_{X_m^c}\Big)^{-1}\Big(\tilde g_1\big(\Pi_{\tilde m}^s DT^{t_1}(x)(y_1^u + y^s + y_1^c)\big) \\
&\quad - \tilde g_2\big(\Pi_{\tilde m}^s DT^{t_1}(x)(y_2^u + y^s + y_2^c)\big) - \Pi_{\tilde m}^c DT^{t_1}(x)(y_1^u - y_2^u)\Big)
\end{aligned}$$

where $\tilde g_1$ and $\tilde g_2$ are Lipschitz representatives of $\tilde\gamma_1(\tilde x)$ and $\tilde\gamma_2(\tilde x)$, respectively, at $\tilde x$ which satisfy (5.2). Using Lemma 3.5, we obtain

$$\begin{aligned}
|y_1^c - y_2^c| &\le \|(\Pi_{\tilde m}^c DT^{t_1}(x)|_{X_M^c})^{-1}\|(\theta|\Pi_{\tilde m}^s DT^{t_1}(x)(y_1^u - y_2^u + y_1^c - y_2^c)| \\
&\quad + \|[\tilde g_1] - [\tilde g_2]\||\Pi_{\tilde m}^s DT^{t_1}(x)(y_2^u + y^s + y_2^c)|) \\
&\quad + o(\Pi_{\tilde m}^s DT^{t_1}(m)(y_2^u + y^s + y_2^c)) + O(\epsilon)|y_1^c - y_2^c| \\
&\le O(\epsilon)|y_1^c - y_2^c| + \big(\lambda + O(\epsilon)\big)\|[\tilde g_1] - [\tilde g_2]\||y^s| + o(|y^s|),
\end{aligned}$$

where $|y_i^c| \le \theta|y^s|$, $|y_i^u| \le \theta(|y^s| + |y_i^c|)$, $i = 1, 2$ are used. Therefore,

$$|y_1^c - y_2^c| \le \frac{\lambda + O(\epsilon)}{1 - O(\epsilon)}\|[\tilde g_1] - [\tilde g_2]\||y^s| + o(|y^s|),$$

which yields

$$\|[g_1] - [g_2]\| \le \frac{\lambda + O(\epsilon)}{1 - O(\epsilon)}\|[\tilde g_1] - [\tilde g_2]\|.$$

Since $0 < \lambda < 1$, for $\lambda < \lambda_1 < 1$, we may choose ϵ^* small enough so that for $\epsilon < \epsilon^*$

$$\frac{\lambda + O(\epsilon)}{1 - O(\epsilon)} < \lambda_1.$$

Thus,

$$\|\mathcal{F}(\tilde\gamma_1) - \mathcal{F}(\tilde\gamma_2)\| \le \lambda_1\|\tilde\gamma_1 - \tilde\gamma_2\|. \tag{5.7}$$

This completes the proof. □

Our goal is to find a unique fixed point of $\mathcal{F}$ in $\Sigma_\theta^{ss,\ell}$. The difficulty here is that $\Sigma_\theta^{ss,\ell}$ may not be a complete space. On the other hand, from (5.7), we have

$$\|\mathcal{F}^k(\tilde\gamma_1) - \mathcal{F}^k(\tilde\gamma_2)\| \le \lambda_1^k\|\tilde\gamma_1 - \tilde\gamma_2\|,$$

which yields that at each $x = m + x^s + f_m^{ss}(x^s) \in W^{cs}(\epsilon)$

$$\mathcal{F}^k(\gamma)(x)$$

is a Cauchy sequence in $J^c(m)$. Since $J^c(m)$ is a Banach space, we have

$$\mathcal{F}^k(\gamma)(x) \to \gamma_0(x)$$

where $\gamma_0(x) \in J^c(X_m^s, X_m^c; 0, 0)$. Clearly, the limit γ_0 is unique and does not depend on the initial choice of γ. We shall show that $\gamma_0 \in \Sigma_\theta^{ss,\ell}$.

For each $x = m + x^s + f_m^{ss}(x^s) \in W^{cs}(\epsilon)$, let

$$x^u + x^c = f_m^{ss}(x^s).$$

Set

$$f_1(y^s) = \Pi_m^c(f_m^{ss}\big(y^s + x^s) - f_m^{ss}(x^s)\big).$$

Then, we have $f_1(0) = 0$ and

$$|f_1(y_1^s) - f_1(y_2^s)| \le \theta |y_1^s - y_2^s|$$

for $y_1^s, y_2^s \in X_m^s(r)$ for some $r > 0$. Thus, f_1 induces $\gamma_1 \in \Sigma_\theta^{ss,\ell}$ by

$$\gamma_1(x) = [f_1].$$

Lemma 5.7. *$\mathcal{F}(\gamma_1) = \gamma_1$ and hence $\gamma_0 = \gamma_1$.*

Proof. For $x = m + x^s + f_m^{ss}(x^s) \in W^{cs}(\epsilon)$, we write

$$T^{t_1}(x) = \tilde{x} = \tilde{m} + \tilde{x}^s + f_{\tilde{m}}^{ss}(\tilde{x}^s).$$

Let

$$\tilde{f}(\tilde{y}_s) = \Pi_{\tilde{m}}^c\big(f_{\tilde{m}}^{ss}\big(\tilde{y}_s + \tilde{x}^s\big) - f_{\tilde{m}}^{ss}(\tilde{x}^s)\big).$$

Let f be given by Lemma 5.2 through $\tilde{f}$. We want to show $[f_1] = [f]$. For $y^s \in X_m^s(r)$ let

$$y_1^c = f_1(y^s),$$
$$y^c = f(y^s).$$

From the definition of f,

$$\begin{aligned} y^c = (\Pi_{\tilde{m}}^c DT^{t_1}(x)|_{X_m^c})^{-1}(\tilde{f}\big(\Pi_{\tilde{m}}^s DT^{t_1}(x)(y_1^u + y^s + y^c)\big) \\ - \Pi_{\tilde{m}}^c DT^{t_1}(x)(y_1^u + y^s)), \end{aligned} \tag{5.8}$$

where $y_1^u = Df_m^{cs}(x^s, x^c)(y^s, y^c)$. On the other hand, from the invariance of $\{f_m^{ss}\}$, there exists $\tilde{y}^s \in X_{\tilde{m}}^s(\tilde{r})$ such that

$$T^{t_1}(x + y^u + y^s + f_1(y^s)) = \tilde{x} + \tilde{y}^u + \tilde{y}^s + \tilde{f}(\tilde{y}^s)$$

where

$$\begin{aligned} y^u &= \Pi_m^u\big(f_m^{ss}(y^s + x^s) - f_m^{ss}(x^s)\big) = f_m^{cs}(x^s + y^s, x^c + y^c) - f_m^{cs}(x^s, x^c), \\ \tilde{y}^u &= \Pi_{\tilde{m}}^u(f_{\tilde{m}}^{ss}(\tilde{y}^s + \tilde{x}^s) - f_{\tilde{m}}^{ss}(\bar{x}^s)). \end{aligned}$$

By the Taylor expansion, we obtain

$$\tilde{y}^u + \tilde{y}^s + \tilde{y}^c = DT^{t_1}(x)(y^u + y^s + y_1^c) + o(|y^u| + |y^s| + |y_1^c|),$$

where $\tilde{y}^c = \tilde{f}(\tilde{y}^s)$.

Note that $|y^u| + |y_1^c| \le \theta |y^s|$. Hence,

$$\tilde{y}^u + \tilde{y}^s + \tilde{y}^c = DT^{t_1}(x)(y^u + y^s + y_1^c) + o(|y^s|)$$

and for $\alpha = u, s, c$,

$$\tilde{y}^\alpha = \Pi_{\tilde{m}}^\alpha DT^{t_1}(x)(y^u + y^s + y_1^c) + o(|y^s|).$$

In particular,

$$\tilde{y}^c = \Pi_{\tilde{m}}^c DT^{t_1}(x)(y^u + y^s + y_1^c) + o(|y^s|),$$

which yields

$$\begin{aligned} y_1^c &= (\Pi_{\tilde{m}}^c DT^{t_1}(x)|_{X_m^c})^{-1}(\tilde{y}^c - \Pi_{\tilde{m}}^c DT^{t_1}(x)(y^u + y^s)) + o(|y^s|) \\ &= (\Pi_{\tilde{m}}^c DT^{t_1}(x)|_{X_m^c})^{-1}(\tilde{f}(\tilde{y}^s) - \Pi_{\tilde{m}}^c DT^{t_1}(x)(y^u + y^s)) + o(|y^s|) \\ &= (\Pi_{\tilde{m}}^c DT^{t_1}(x)|_{X_m^c})^{-1}(\tilde{f}(\Pi_{\tilde{m}}^s DT^{t_1}(x)(y^u + y^s + y_1^c)) \\ &\qquad - \Pi_{\tilde{m}}^c DT^{t_1}(x)(y^u + y^s)) + o(|y^s|). \end{aligned}$$

Using (5.8) and Lemma 3.5, we obtain

$$|y^c - y_1^c| \le O(\epsilon)|y^c - y_1^c| + o(|y^s|).$$

Thus, by choosing ϵ^* small enough, we obtain

$$|y^c - y_1^c| \le o(|y^s|),$$

that is,

$$|f_1(y^s) - f(y^s)| \le o(|y^s|)$$

and hence,

$$[f_1] = [f].$$

The uniqueness of the limit γ_0 implies $\gamma_0 = \gamma_1$. This completes the proof. □

We are now ready to show

Proposition 5.8. *$f_m^{ss}(y^s)$ is C^1 in y^s and $Df_m^{ss}(y^s)$ is uniformly continuous in m.*

Proof. It is clear that from the definition of $\mathcal{F}$, particularly the definition of E, we have that if $\gamma \in \Sigma_\theta^{ss,a}$, then $\mathcal{F}(\gamma) \in \Sigma_\theta^{ss,a}$. Since $J^a(m) = J^d(m)$ is closed, $\gamma_0(x) = \gamma_0(m + x^s + f_m^{ss}(x^s)) \in J^a(m)$, where γ_0 is the limit of the Cauchy sequence $\mathcal{F}^k(\gamma)$. Since $\gamma_0 = \gamma_1$ is unique, for $x = m + x^s + f_m^{ss}(x^s)$, we have $[f_1] \in J^a(m)$, that is, $\Pi_m^c f_m^{ss}(y^s)$ is differentiable at x^s, which together with the fact that f_m^{cs} is C^1 yields that f_m^{ss} is differentiable.

Finally, we want to show that $Df_m^{ss}(x^s)$ is continuous in x^s. It is enough to show that $D\Pi_m^c f_m^{ss}(x^s)$ is continuous.

Let $L(X^s, X^c)$ denote the vector bundle over $W^{cs}(\epsilon)$ with fiber $L(X_m^s, X_m^c)$ on $W^{cs}(\epsilon) \cap W_m^{ss}(3\epsilon)$, where $L(X_m^s, X_m^c)$ is the Banach space of all bounded linear operators from X_m^s to X_m^c.

Define the space

$$\Lambda_\theta^{ss} = \Big\{\ell : W^{cs}(\epsilon) \to L(X^s, X^c) \text{ is a } C^0 \text{ section and } \|\ell\| \le \theta\Big\}$$

with the norm

$$\|\ell\| = \sup\Big\{||\ell(x)|| : x \in W^{cs}(\epsilon)\Big\}.$$

It is easy to check that Λ_θ^{ss} is a complete metric space. The space Λ_θ^{ss} may be regarded as a subset of $\Sigma_\theta^{ss,\theta}$ by identifying ℓ and $[\ell]$. As in [BLZ1], we have $\mathcal{F} : \Lambda_\theta^{ss} \to \Lambda_\theta^{ss}$ is a contraction and has a unique fixed point $\ell \in \Lambda_\theta^{ss}$. The uniqueness of γ_0 implies

$$\gamma_0 = [\ell],$$

which yields that Df_1 is continuous. The proof of the uniform continuity of $Df_m^{ss}(y^s)$ in m follows in the same fashion as in Lemma 4.5. The proof is complete. □

For $m \in M$, define

$$W_m^{ss}(\epsilon) = \{m + x^s + f_m^{ss}(x^s) \in W^{cs}(\epsilon) : \ |x^s| < 3\epsilon\},$$

i.e., $W_m^{ss}(\epsilon)$ is the intersection of the center-stable manifold of size ϵ and the stable fiber at m.

The next result gives a geometric structure of the stable fibers.

Proposition 5.9. *Let* $Q_{m_0} = \{x^s \in X^s_{m_0}(3\epsilon) : \ m_0 + x^s + f^{ss}_{m_0}(x^s) \in W^{cs}(\epsilon)\}$. *Then* Q_{m_0} *is star-shaped.*

Proof. For any $x^s \in X^s_{m_0}$ with $|x^s| = 1$ and $\tau \in [0, 3\epsilon]$, we write

$$m_0 + \tau x^s + f^{ss}_{m_0}(\tau x^s) = m_\tau + \Pi^u_{m_\tau} x^u_\tau + \Pi^s_{m_\tau} x^s_\tau, \tag{5.9}$$

where $x^\alpha_\tau \in X^\alpha_{m_0}$ for $\alpha = u, s$. It is enough to show that $|\Pi^s_{m_\tau} x^s_\tau|$ is increasing in τ. Computing the derivative of (5.9) with respect to τ, we have

$$\begin{aligned} x^s + Df^{ss}_{m_0}(\tau x^s)x^s = \bar{y}^c &+ D\Pi^u_{m_\tau}(\bar{y}^c)x^u_\tau \\ &+ D\Pi^s_{m_\tau}(\bar{y}^c)x^s_\tau + \Pi^u_{m_\tau} y^u + \Pi^s_{m_\tau} y^s, \end{aligned} \tag{5.10}$$

where $\bar{y}^c = \frac{dm_\tau}{d\tau} \in X^c_{m_\tau}$ and $y^\alpha = \frac{dx^\alpha_\tau}{d\tau} \in X^\alpha_{m_0}$ for $\alpha = u, s$. Note that

$$||Df^{ss}_{m_0}(\tau x^s)|| < \mu.$$

Applying $\Pi^s_{m_0}$, $\Pi^u_{m_0}$, and $\Pi^c_{m_0}$ to (5.10), respectively, we have

$$y^s = x^s + O(\epsilon)(\bar{y}^c + y^u + y^s),$$

$$|y^u| < \mu|x^s| + O(\epsilon)(|\bar{y}^c| + |y^s|),$$

and

$$|\bar{y}^c| < \mu|x^s| + O(\epsilon)(|y^u| + |y^s|).$$

Therefore,

$$y^s = x^s + O(\epsilon)x^s.$$

Thus,

$$\begin{aligned} (\Pi^s_{m_\tau} x^s_\tau)' &= D\Pi^s_{m_\tau}(\bar{y}^c)x^s_\tau + \Pi^s_{m_\tau} y^s \\ &= x^s + O(\epsilon)x^s \\ &= \Pi^s_{m_\tau} x^s_\tau + O(\epsilon)\Pi^s_{m_\tau} x^s_\tau, \end{aligned}$$

which yields, by using the Taylor expansion, that $|\Pi^s_{m_\tau} x^s_\tau|$ is increasing in τ. The proof is complete. □

Summarizing the results we have obtained so far, gives Theorem 2.2.

6. C^1 UNSTABLE FOLIATION

Generally, a few modifications are needed to extend our results in Sections 4 and 5 to the case of $W^{cu}(\epsilon)$. Here we present an outline of these modifications. The most significant differences are the definition of the graph transform and the associated spaces. We shall leave the details to the interested reader.

Corresponding to the idea of stable fiber, we define the following:

Definition. Let $x = m + x^u + x^s \in \Theta(X^u(\epsilon) \oplus X^s(\epsilon))$. A map $f_x : X^u_m(3\epsilon) \to X^s_m(\epsilon) \oplus X^c_m(\epsilon)$ is called an unstable fiber if

(1) $f_x(0) = 0$;

(2) f_x is Lipschitz continuous with Lipschitz constant $\mathrm{Lip}(f_x) < \mu$.

We shall construct an unstable invariant foliation in the following space

$$\Gamma_\epsilon^{uu} = \Big\{ F = \{f_x\}_{x \in \overline{W^{cu}(\epsilon)}} : f_x \text{ is an unstable fiber and}$$
$$\text{is uniformly continuous in } x \Big\}.$$

In order to establish Theorem 2.3, we first construct an invariant foliation with Lipschitz fibers, then show the fibers are C^1 and uniformly continuous with respect to base points. The outline of the approach is as follows: The first step is to construct $\tilde{F} \in \Gamma_\epsilon^{uu}$ for each $F \in \Gamma_\epsilon^{uu}$ such that $\tilde{F}$ is the image of F under the time-t_1 map T^{t_1} in a certain sense. This defines a graph transform

$$\mathcal{F}(F) = \tilde{F}.$$

In the case of the stable foliation, we constructed a preimage $\tilde{F}$ of F instead of the image under T^{t_1}.

The next step is to show that $\mathcal{F}$ is a contraction in Γ_ϵ^{uu} under the norm

$$||F|| = \sup_{x \in \overline{W^{cu}(x)}} ||f_x||$$

where

$$||f_x|| = \sup_{y^u \in X^u(3\epsilon), y_u \neq 0} \frac{|f_x(y^u)|}{|y^u|}.$$

Thus, $\mathcal{F}$ has a unique fixed point $F^{uu} = \Big\{ f_x^{uu} \Big\} \in \Gamma_\epsilon^{uu}$.

The third step is to establish properties analogous to those for the stable foliation.

The final step is to show that each unstable fiber f_m^{uu} is C^1 and continuous in m.

Let us first look at how to construct $\mathcal{F}$. Let $\tilde{x} \in \Theta(X^u(\epsilon) \oplus X^s(\epsilon))$ be such that there exists $x \in \Theta(X^u(\epsilon) \oplus X^s(\epsilon))$ satisfying $\tilde{x} = T^{t_1}(x)$. For a given unstable fiber

$$f_x : X_m^u(3\epsilon) \to X^s(\epsilon) \oplus X^c(\epsilon),$$

we first want to construct an unstable fiber $\tilde{f}_{\tilde{x}}$ at $\tilde{x}$ such that

$$T^{t_1}(\text{graph}(f_x)) \supset \text{graph}(\tilde{f}_{\tilde{x}}).$$

To do this, for each $\tilde{y}^u \in X_{\tilde{m}}^u(3\epsilon)$, we define a map A from $X_m^u(3\epsilon)$ to X_m^u by

$$A(y^u) = \left(\Pi_{\tilde{m}}^u DT^{t_1}(m_1)|_{X_m^u}\right)^{-1} \Big(\tilde{y}^u - \Pi_{\tilde{m}}^u(T^{t_1}(x + y^u + f_x(y^u)) - \tilde{x})\Big) + y^u.$$

In the same fashion as for A in Lemma 4.3, one may show that there exists $\epsilon^* > 0$ such that if $\epsilon < \epsilon^*$, then A is a contraction from $\overline{X_m^u(3\epsilon)}$ into $X_m^u(3\epsilon)$, hence, has a unique fixed point $y^u \in X_m^u(3\epsilon)$. For $\alpha = u, s, c$ let

$$\bar{y}^\alpha = \Pi_{\tilde{m}}^\alpha(T^{t_1}(x + y^u + f_x(y^u)) - \tilde{x}).$$

Note that $\tilde{y}^u = \bar{y}^u$. Define

$$\tilde{f}_{\tilde{x}}(\bar{y}^u) = \bar{y}^s + \bar{y}^c.$$

Then, $\tilde{f}_{\tilde{x}}$ is an unstable fiber with Lipschitz constant $\lambda_1\mu$. Let $F \in \Gamma_\epsilon^{uu}$. For $\tilde{x} \in W^{cu}(\epsilon)$, by the negative invariance property of $W^{cu}(\epsilon)$, there exists a unique $x \in W^{cu}(\epsilon)$ such that $T^{t_1}(x) = \tilde{x}$. The above discussion produces a unique unstable fiber $\tilde{f}_{\tilde{x}}$ at $\tilde{x}$ such that

$$T^{t_1}(\text{graph}(f_x)) \supset \text{graph}(\tilde{f}_{\tilde{x}}).$$

Let $\tilde{F} = \left\{ \tilde{f}_{\tilde{x}} : \tilde{x} \in W^{cu}(\epsilon) \right\}$. Thus, one may define a graph transform $\mathcal{F}$ from Γ_ϵ^{uu} into itself by

$$\mathcal{F}(F) = \tilde{F}.$$

Using a similar argument to that used in Lemma 4.6, one can show that $\mathcal{F}$ is a contraction, hence has a unique fixed point $F^{uu} \in \Gamma_\epsilon^{uu}$. Propositions 4.7–4.11 carry over with a little modification in the case of the unstable foliation. To show f_m^{uu} is C^1, we define

$$\Sigma_\theta^{uu,l} = \left\{ \gamma : W^{cu}(\epsilon) \to J_\theta^l(X^u, X^c; 0, 0) \text{ with } ||\gamma|| < \infty \right\}.$$

In order to show the smoothness of f_m^{uu}, we need to find a candidate for the tangent bundle of f_m^{uu}, then to show it indeed is the tangent bundle. The approach is the same as that for the smoothness of the center unstable manifold in [BLZ1], which is modified from the stable case. The significant difference is the graph transform. For each $\gamma \in \Sigma_\theta^{uu,\ell}$, we want to construct $\tilde{\gamma} \in \Sigma_\theta^{uu,\ell}$ such that $\tilde{\gamma}$ is the image of γ under DT^{t_1} in a certain sense, defining a graph transform

$$\mathcal{F}(\gamma) = \tilde{\gamma}.$$

For each $\tilde{x} = \tilde{m} + \tilde{x}^u + \tilde{x}^s + \tilde{x}^c \in W^{cu}(\epsilon)$, where $\tilde{x}^s + \tilde{x}^c = f_{\tilde{m}}^{uu}(\tilde{x}^u)$, by Theorem A, there exists a unique point $x = m + x^u + x^s + x^c \in W^{cu}(\epsilon)$, where $x^s + x^c = f_{\tilde{m}}^{uu}(x^u)$, such that $\tilde{x} = T^{t_1}(x)$. For $\gamma \in \Sigma_\theta^{uu,\ell}$, let g be a Lipschitz representative of $\gamma(x)$. We want to construct a Lipschitz map

$$\tilde{g} : X_{\tilde{m}}^u(\tilde{r}) \to X_{\tilde{m}}^c(\tilde{r})$$

such that for $\tilde{y}^u \in X_{\tilde{m}}^u(\tilde{r})$, there exists a unique $y^u \in X_m^u(r)$ such that

$$DT^{t_1}(x)(y^u + y^s + g(y^u)) = \tilde{y}^u + \tilde{y}^s + \tilde{g}(\tilde{y}^u)$$

where,

$$y^s = Df_m^{cu}(x^u, x^c)(y^u, g(y^u)),$$
$$\tilde{y}^s = Df_m^{cu}(\tilde{x}^u, \tilde{x}^c)(\tilde{y}^u, \tilde{g}(\tilde{y}^u)).$$

To do this, for each $\tilde{y}^u \in X_{\tilde{m}}^u(\tilde{r})$, we define a map E from $\overline{X_m^u(r)}$ to X_m^u by

$$E(y^u) = \left(\Pi_{\tilde{m}}^u DT^{t_1}(m)\Big|_{X_m^u} \right)^{-1} \Big(\tilde{y}^u - \Pi_{\tilde{m}}^u DT^{t_1}(x)\big(y^u + y^s + g(y^u)\big) \Big) + y^u$$

where $y^s = Df_m^{cu}(x^u, x^c)\big(y^u, g(y^u)\big)$. In the same fashion as for E in Section 5, one may show that for small r, E is well defined and a contraction from $\overline{X_m^u(r)}$ into $X_m^u(r)$. Thus, E has a unique fixed point y^u. Let

$$\tilde{y}^c = \Pi_{\tilde{m}}^c DT^{t_1}(x)\big(y^u + y^s + g(y^u)\big).$$

Define

$$\tilde{y}^c = \tilde{g}(\tilde{y}^u).$$

Then, $\tilde{g}$ satisfies the desired conditions. Thus, one can define $\tilde{\gamma}$ by

$$\tilde{\gamma}(\tilde{x}) = [\tilde{g}].$$

The induced graph transform is given by

$$\mathcal{F}(\gamma) = \tilde{\gamma}.$$

The remaining arguments follow in the same way as those in Section 5.

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